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A REEXAMINATION OF THE COASE THEOREM

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ABSTRACT

This paper makes three advances: 1) It fixes the empty-core problem of the Coase theorem; 2) it provides the smallest upper bound of transaction costs below which the optimal or efficient outcomes can be achieved; and 3) it establishes two mathematical theorems that capture the main insights and major aspects of the Coase theorem. A simpler version of the theorems says that in a coalitional production economy without transaction costs, the maximal payoff will be produced by the optimal firms and be allocated in the always non-empty core.

Keywords: Coalition formation, core, optimal firms.
JEL Classification Numbers: C71, D23, L11, L23.

1. INTRODUCTION

The Coase theorem is one of the most cited, most debated and most confusing "theorems" in economics.[1] One popular version is stated as: Bargaining will lead to a Pareto efficient outcome if transaction costs are sufficiently low,

This paper replaces earlier drafts circulated under various titles, such as "The maximal payoff and coalition formation in coalitional games" (2006) and "Two mathematical versions of the Coase theorem" (2016), which had been cited in several previous studies (e.g., Arribillaga, 2010; Cesco, 2012). I thank Donald Smythe, Hao Wang, Zaifu Yang, two anonymous referees, and seminar/conference participants at Huazhong U of Sciences and Technology, Institute of System Sciences (CAS), Konan U, Kyoto U, Peking U, Renmin U of China, U of Saskatchewan, Waseda U, CEA Annual Conference, Canadian Econ Theory Conference, Coalition Theory Network Workshop, and World Congress of Game Theory Society for valuable comments and encouragements. All errors, of course, are my own.

[1] The earliest reference to (or version of) Coase theorem was in Stigler (1966, 113): "The Coase theorem thus asserts that under perfect competition private and social costs will be equal."

regardless of the initial allocation of property. As in the claim by Regan (1972, 428) that "the Coase theorem is a proposition in the theory of games, and not a proposition about traditional markets or competitive equilibrium," this paper formalizes the Coase theorem as two mathematical theorems in cooperative games and elevates it to an economic law on non-market transactions that is comparable to Smith's invisible hand on market transactions.

Precisely, this paper advances the literature on the Coase theorem in three aspects: (i) It fixes the purported empty-core problem of the Coase theorem (Aivazian & Callen, 1981; Coase, 1981); (ii) it provides the smallest upper bound of transaction costs below which the optimal or efficient outcomes can be achieved; and (iii) it establishes two mathematical theorems that capture the main insights and major aspects of the Coase theorem. The mathematical essence of these two theorems is the duality between sub-coalitions' producing ability (which defines the maximal payoff) and blocking power (which defines the core stability). Such duality reveals that a firm's shadow value is equal to its balancing weight, and it implies not only the three known theorems for a non-empty usual core but also three new core theorems in coalitional games.

The success in the author's search for a repair of the empty-core problem is the discovery that maximal profits in the example of Aivazian & Callen (1981), or efficient outcomes in more general cases, are achieved in advanced forms of production or minimal balanced collections of the firms rather than conventional forms of production such as the monopoly or other partitions of the firms.

The rest of the paper is organized as follows. Section 2 provides a fix for the empty-core problem in the example of Aivazian & Callen (1981). Section 3 studies the duality between sub-coalitions' blocking power and producing ability in coalitional TU (transferable utility) games. Sections 4 and 5 establish the TU and NTU (non-transferable utility) Coase theorems, respectively. Section 6 concludes, and the appendix provides proofs.

2. A FIX FOR THE PURPORTED EMPTY-CORE EXAMPLE OF THE COASE THEOREM

An important study of the Coase theorem was the following empty-core example reported in Aivazian & Callen (1981):

Example 2.1 *There are seven possible firms:*[2] $S = 1, 2, 3, 12, 13, 23,$ *and* 123, *and their daily profits are:* $v(1) = 3000$, $v(2) = 8000$, $v(3) = 24000$; $v(12) = 15000$, $v(13) = 31000$, $v(23) = 36000$; *and* $v(123) = 40000$.

This is an example of three-person coalitional TU game or three-owner coalitional economy. Its five partitions or conventional market structures are: $B_0 = \{1,2,3\}$, $B_1 = \{1,23\}$, $B_2 = \{2,13\}$, $B_3 = \{3,12\}$, and $B_m = \{123\}$, whose total profits satisfy: $\pi(B_0) = v(1) + v(2) + v(3) = 35000 < \pi(B_1) = v(1) + v(23) = \pi(B_2) = \pi(B_3) = 39000 < \pi(B_m) = v(123) = 40000$. Since a monopoly produces the largest profits among the five partitions, the Coase theorem implies that the monopoly or monopoly merger will be formed. However, this conclusion breaks down after one checks coalitional rationality. Let $x = (x_1, x_2, x_3) \geq 0$ be a split of $\pi(B_m) = 40000$. Rationality requires that the split has no blocking coalitions, or be in the usual core, or satisfy the following two sets of inequalities:

i) $x_1 \geq v(1) = 3000$, $x_2 \geq v(2) = 8000$, $x_3 \geq v(3) = 24000$;

ii) $x_1 + x_2 \geq v(12) = 15000$, $x_1 + x_3 \geq v(13) = 31000$, and $x_2 + x_3 \geq v(23) = 36000$.

Adding up the inequalities in (*ii*) yields $x_1 + x_2 + x_3 \geq (15000 + 31000 + 36000)/2 = 41000 > 40000$, which contradicts $x_1 + x_2 + x_3 = 40000$. Thus any split of monopoly profits is blocked by at least one coalition. Consequently, the monopoly cannot be formed so the Coase theorem fails.

Is the above argument false, or could the insight of the Coase theorem possibly be wrong? Telser (1994) observed that "Coase's elaborate analysis in his comment (1981) fails to come to grip with the issues raised by this example," and concluded that "The Coase "theorem" needs much repair when there is an empty core" (see Aivazian & Callen, 2003, for a discussion). Such a serious issue remained unsettled for more than three decades, until this study. To see the repair, move a step deeper inside this coalitional economy, assume that each of the three singleton firms is owned by a full-time worker with 8 hours of labor inputs, and that they produce a homogeneous product called profit from labor by the following linear production functions:

$f_1(x) = 3000x/8 = 375x$, $f_2(x) = 1000x$, $f_3(x) = 3000x$, $0 \leq x \leq 8$;

[2] We simplify $v(\{i\})$ as $v(i)$, $v(\{1,2\})$ as $v(12)$. Similar simplifications apply to other coalitions.

$f_{12}(x) = 937.5x$, $f_{13}(x) = 1937.5x$, $f_{23}(x) = 2250x$, $0 \leq x \leq 16$; and $f_{123}(x) = 5000x/3$, $0 \leq x \leq 24$.

The singleton, two-member and monopoly firms have 8, 16 and 24 hours of fixed labor inputs, respectively. It is easy to see that these production functions generate the same profits as those in Example 2.1. Now, consider operating each of the three two-party firms at full capacity for 4 hours (or at half capacity for 8 hours), and assume such re-organization of production does not involve any new costs (i.e., zero transaction cost). This can be arranged, for example, in the following sequence: $S = 12$ opens at full capacity from 8:00 *a.m.*-noon, $S = 13$ from noon-4:00 *p.m.*, and $S = 23$ from 4:00-8:00*p.m.* The profits from this new or advanced form of production are truly maximal and are given by $mp = f_{12}(8) + f_{13}(8) + f_{23}(8) = [v(12) + v(13) + v(23)]/2 = 41000 > \pi(B_m) = 40000$, which exceeds the profits of operating the monopoly at full capacity for 8 hours. Define the new core as the splits of the above mp that are unblocked by all subcoalitions. One can check that the new core of Example 2.1 has a unique vector: $x_1 = 5000$, $x_2 = 10000$, $x_3 = 26000$, which is the optimal outcome predicted by the Coase theorem.[3]

Note that the above optimal outcome is achieved when all three firms open for business only half of the time and all three workers have two part-time jobs, instead of one full-time job. This conclusion, not reported in previously literature, will provide a new argument in studying labor theory and production theory.

Now, the repair is to replace conventional forms of production with advanced forms of production, or replace the usual core with the always non-empty new core (or simply, the core), so the Coase theorem is now free of the empty-core problem and will be precisely stated as Theorem 2 (with transferable utility) in section 4 and Theorem 4 (with non-transferable utility) in section 5.

[3] Related empty-core problems can be similarly fixed. Consider, for example, the empty-core in Gonzalez et al. (2016), which is caused by adding the democratic and fairness properties in their study of efficiency and neutrality (Calabresi, 1968; Regan, 1972). By replacing the grand coalition's payoff in their equation (3) with the game's maximal payoff (i.e., replacing the usual core with the new core), efficiency will always hold, and the Coase theorem will be free of the empty-core problem.

3. THE MAXIMUM OF GENERATED-PAYOFFS AND THE DUALITY IN COALITIONAL TU GAMES

Let $N = \{1, 2, \ldots, n\}$ be the set of players, $\mathbb{N} = 2^N$ be the set of all coalitions. A coalitional TU game (or a game in characteristic form) is given by

$$\Gamma = \{N, v(\cdot)\}, \tag{1}$$

which is a set function $v : \mathbb{N} \to R_+$ with $v(\varnothing) = 0$, specifying a joint payoff $v(S)$ for each coalition $S \subseteq N$. We will refer game (1) as a coalitional economy and a player i as the owner of firm i, or as a worker i, when the emphasis is on the Coase theorem. We use a lowercase v in $v(\cdot)$ to define the above TU game (1), and an uppercase V in $V(\cdot)$ to define NTU games in section 5.

Let $X(v(N) = \{x \in R^n_+ | \Sigma_{i \in N} x_i = v(N)\}$ denote the preimputation space or the set of payoff vectors that are splits of $v(N)$. A split $x = (x_1, \ldots, x_n) \in X(v(N))$ satisfies the rationality of a coalition $S \subseteq N$ or is unblocked by S if it gives S no less than $v(S)$ (i.e., $\Sigma_{i \in S} x_i \geq v(S)$), and is in the usual core[4] if it is unblocked by all $S \neq N$. Denote the usual core of (1) as

$$c_0(\Gamma) = \{x \in X(v(N)) | \Sigma_{i \in S} x_i \geq v(S) \text{ for all } S \neq N\}. \tag{2}$$

We use a lowercase c in $c_0(\Gamma)$ to denote the above TU core and an uppercase C in $C_0(\Gamma)$ to denote the NTU core in section 5.

The concept of generated-payoffs is defined by balanced collections of coalitions. Given a collection of coalitions $B = \{T_j \subseteq N | j = 1, \ldots, k\}$ and a player $i \in N$, let $B(i) = \{T \in B | i \in T\}$ denote the subset of coalitions of which i is a member. Then, B is a balanced collection if it has a balancing vector $w = \{w_T | T \in B\} \in R^k_{++}$ such that $\Sigma_{T \in B(i) W_T} = 1$ for each $i \in N$.

To see the intuition of a balancing vector, treat game (1) as a coalitional economy as in Example 2.1, where each singleton firm i has one full-time worker or 8 hours of labor inputs. Thus, each firm $S \subseteq$ N will have $8k(S)$ hours of labor inputs or $k(S)$ workers (i.e., $k(S) = |S|$ is the cardinality of S), and

[4] Note that Shapley (1955), not Gillies (1959), was the earliest known paper that defined the core. Zhao (2018) has summarized the early history of the core, from six sources, as: *i*) Gillies first used the term core during 1952-1953 referring to some intersections of the stable sets, *ii*) Shapley first formulated core solution during a conversation with Shubik in the same two-year period, and *iii*) Shapley presented his core solution in Kuhn's conference (March 1953), three months before Gillies finished his dissertation in June 1953.

produces a daily profit of $v(S)$ by operating at full capacity for 8 hours, with the following linear production function:

$$f_S(x) = v(S)x/[8k(S)], 0 \leq x \leq 8k(S). \tag{3}$$

Now, each balanced B with a balancing vector w defines the following form of production:

Operate each firm $T \in B$ at full capacity for $8w_T$ hours, and produce a profit equal to

$$gp(B) = \Sigma_{T \in B} f_T(8w_T k(T)) = \Sigma_{T \in B} w_T v(T). \tag{4}$$

The above operation and its generated payoff $gp(B)$ are feasible because the condition $\Sigma_{T \in B(i)} w_T = 1$ or balancedness ensures that each worker i works exactly 8 hours (i.e., i works for $8w_T$ hours at each T in $B(i)$). In Example 2.1, the collection $B_5 = \{12, 13, 23\}$ with balancing vector $w = (1/2, 1/2, 1/2)$ yields the payoff $gp(B_5) = [v(12) + v(13) + v(23)]/2 = 41000$.

Thus, balancing weights here are proportions of inputs that each firm $T \in B$ receives from all its owners (or that a worker i allocates to his/her firms in $B(i)$).[5] Such productions represent an advance beyond conventional forms of production or partitions, because they include partitions as special cases and they could, as in Example 2.1, possibly produce higher payoffs. The discovery of such higher payoffs beyond $v(N)$ is the key in our search for a repair for the empty-core problem of the Coase theorem.[6]

A balanced collection is minimal if no proper subcollection is balanced. It is known that a balanced collection is minimal if and only if its balancing vector is unique (Shapley, 1967). Denote the set of all minimal balanced collections (excluding the grand coalition) as

$$\Phi = \{B = \{T_1, \ldots, T_k\} | N \notin B, B \text{ is a minimal balanced collection}\}. \tag{5}$$

[5] So far we have three interpretations of the balancing weights: *i*) percentage of time during which the firm operates; *ii*) proportion of inputs that the firm receives; and *iii*) the frequency (or probability) with which the firm forms (or a player joins his/her coalition), assuming that the game is replicated/repeated for a finite number of times (or that uncertainty is added into the game). Other interpretations remain to be discovered.

[6] It is known that the grand coalition's payoff $v(N)$ might be less than the total payoffs in a non-monopoly partition or in a balanced collection, see Sun et al. (2008) on market games, Bennett (1983) on aspiration-core, and Guesnerie & Oddou (1979) on c-core. However, this study is the first to connect with the Coase theorem and treat it as a situation in which non-monopoly firms could produce more than the monopoly does.

In three-person games like Example 2.1, Φ has five entries: the four non-monopoly partitions, B_0, B_1, B_2, B_3, plus $B_5 = \{12, 13, 23\}$. We are now ready to define the maximum of generated-payoffs (mgp) and maximal payoffs (mp):

Definition 3.1 *Given game (1), $gp(B)$ and Φ in (4)-(5), mgp and mp are given by*

$$mgp = mgp(\Gamma) = Max\{gp(B)|B \in \Phi\}, \text{ and} \tag{6}$$
$$mp = mp(\Gamma) = Max\{mgp(\Gamma),\ v(N)\}. \tag{7}$$

The definition considers only minimal balanced collections because mgp is achieved among minimal balanced collections, just as the optimal value in linear programming is achieved among the extreme points.[7]

The following duality result is the theoretical foundation of our repair for the empty-core problem of the Coase theorem.

Theorem 3.1 *Given game (1), the maximization problem (6) is dual to the following minimization problem for the minimum no-blocking payoff (mnbp Zhao, 2001):*

$$mnbp = mnbp(\Gamma) = Min\{\Sigma_{i\in N} x_i | \Sigma_{i\in S} x_i \geq v(S), all\ S \neq N\}, \tag{8}$$

so $mgp(\Gamma) = mnbp(\Gamma)$ holds.

By the above duality, a firm's shadow value in (8) is equal to the balancing weight in (6), and the minimal worth of the grand coalition needed to guarantee no-blocking given in (8) is equal to the maximal payoffs produced by sub-coalitions given in (6). Because *mnbp* represents sub-coalitions' power to block proposed splits of $v(N)$ (Zhao, 2001),[8] mgp represents their ability to produce payoffs that are different from $v(N)$, their producing ability and blocking power are dual to each other.

Note that the above linear programming problem (8) is different from earlier ones for proving the Bondareva-Shapley theorem (e.g., Bondareva 1962;

[7] It is easy to show that the above mp is equal to the grand coalition's payoff in the game's cover. When functions in (3) satisfy $f_S(x) = \Sigma_{i\in S} f_i(\lambda_i)$ for each S, all $\lambda \in R^S_+$ and $\Sigma_{i\in S}\lambda_i = x$, game (1) becomes identical to a market (T, G, A, U) as defined in Shapley & Shubik (1969), with $T = N$, $G = R^n_+$, $A = I = I_n$, and $U = \{f_i | i \in N\}$.

[8] See Watanabe & Matsubayashi (2013) and Karsten & Basten (2014) for applications of *mnbp*.

Shapley 1967; Myerson 1991, 432) in that the grand coalition's constraint is removed in (8) (i.e., all $S \neq N$) while it was included in earlier ones (i.e., all $S \subseteq N$). Needless to say, it is the removal of the grand coalition's constraint in Zhao (2001) and in this study that distinguishes the author's approach from earlier studies, which leads to the discovery of the duality between the producing ability and the blocking power of all sub-coalitions. Such a duality is perhaps the most salient property in cooperative games, because it has three important advantages as discussed below.

First, it also holds in NTU games (see Theorem 3 in section 5). Second, it implies three existence theorems on the usual core (two known and one new), as given in the next corollary.

Corollary 3.2 *Given game* (1)*, the following four statements are equivalent: i) its usual core is non-empty; ii) the game is balanced (Bondareva, 1962; Shapley, 1967); iii) the grand coalition's payoff is large enough to guarantee no-blocking (Zhao, 2001); and iv) players cannot produce a higher payoff than the grand coalition's payoff.*

Precisely, the above core arguments in parts *ii-iv*) are: *ii*) $\Sigma_{T \in B} w_T v(T) \leq v(N)$ for each balanced B with a balancing vector w, *iii*) $v(N) \geq mnbp(\Gamma)$, and *iv*) $mgp(\Gamma) \leq v(N)$.

Third, Theorem 1 also implies, as shown in the next section, the answers to four important questions arising from the Coase theorem: What payoffs will be split? How will the payoff be split? What firms will form? and How much inputs will each of the formed firms receive? Note that the Coase theorem won't be complete if any of these four questions is not answered.

4. THE TU COASE THEOREM

By the new core argument in part *iv*) of Corollary 1, players in games with an empty usual core will not split the grand coalition's payoff $v(N)$, which is smaller than the game's mgp. Then, what payoffs will they split? We postulate that they split the maximal payoff $mp(\Gamma)$ in (7). By $mp(\Gamma) = v(N \geq mgp(\Gamma)$ if $c_0(\Gamma) \neq \varnothing$, and $mp(\Gamma) = mgp(\Gamma) > v(N)$ if $c_0(\Gamma) = \varnothing$, it stands to reason that they will always split $mp(\Gamma)$. This answers the question of what payoffs will be split.

Next, consider the question of how to split the maximal payoff. Let the set of minimal solutions for $mnbp(\Gamma)$ in (8) be denoted by

$$Y = Y(\Gamma) = Arg - Min\{\Sigma_{i\in N}x_i | \Sigma_{i\in S}x_i \geq v(S),\ \text{all}\ S \neq N\}. \tag{9}$$

Coalitional rationality leads to the new core or simply the core given by $c(\Gamma) =$

$$\{x \in X(mp) | \Sigma_{i\in S}x_i \geq v(S), \text{all}\ S \subseteq N\} = \begin{cases} c_0(\Gamma)\ \text{if}\ v(N) = mp(\Gamma) \\ Y(\Gamma)\ \text{if}\ v(N) < mp(\Gamma), \end{cases} \tag{10}$$

which answers the question of how to split $mp(\Gamma)$. Note that the above core is always nonempty because it is identical to the usual core in (2) when the usual core is nonempty, and the set of minimal solutions in (8) or (9) when the usual core is empty.

It is useful to note that the above core in (10) includes the usual core of the game's cover (or balanced cover, Shapley, 1967), which is a new game $\Gamma_{bc} = \{N, \overline{v}(\cdot)\}$, where each $\overline{v}(S) = mp(\Gamma_S)$ is the maximal payoff of the subgame $\Gamma_S = \{S, v(\cdot)\}$.[9]

Now, consider the question of what firms will form. Because players always split mp, they will form the optimal collections that generate mp. Let the set of optimal solutions for $mgp(\Gamma)$in (6) be given by

$$B_0 = B_0(\Gamma) = \{B \in \Phi | gp(B) = mgp(\Gamma)\} = Arg - Max\{gp(B) | B \in \Phi\}. \tag{11}$$

Then, the set of optimal collections can be given as

$$B^* = B^*(\Gamma) = \begin{cases} \{N\}\ \text{if}\ mgp(\Gamma) < v(N) \\ B_0(\Gamma)\ \text{if}\ mgp(\Gamma) > v(N) \\ \{N\} \cup B_0(\Gamma)\ \text{if}\ mgp(\Gamma) = v(N), \end{cases} \tag{12}$$

which answers the question of what firms will form.

Finally, the unique balancing vector for each optimal B answers the question of how much inputs will each of the formed firms receive. The above answers cover all the relevant aspects of the Coase theorem for our coalitional economy (1), which can now be stated as:

[9] The author thanks the referee for pointing out that the core in (10) is identical to the aspiration-core (Bennett, 1983). Because the duality in Theorem 1 yields a new understanding about the internal structure of the core in (10), and because the aspiration-core does not extend to NTU games and is not related to the Coase theorem, the term "core" or "new core" is kept for the set of solutions given in (10).

Theorem 4.1 *In coalitional economy (1) without transaction costs, owners will produce the maximal payoff by allocating inputs to optimal firms based on the firms' shadow values, and they will split the maximal payoff within the non-empty core.*

Precisely, the maximal payoff, core and optimal firms are respectively given in (7), (10), and (12), and the balancing vector for each set of optimal firms in (12) specifies their shadow values or proportions of inputs received from the owners. In Example 2.1, one has $mp(\Gamma) = mgp(\Gamma) = 41000 > v(N) = 40000$. By Theorem 2, the optimal collection or the set of optimal firms with input allocations (or operating hours) is $B^*(\Gamma) = B_0(\Gamma) = \{12, 13, 23\}$ with $w = (1/2, 1/2, 1/2)$, and the core is $c(\Gamma) = Y(\Gamma) = (5000, 10000, 26000)\}$.

One advantage of Theorem 2 is that it provides a method to estimate the bounds of transaction costs (or merging costs) for each firm or coalition S, using the approach introduced by the author (Zhao, 2009). For simplicity, let the merging costs for monopoly be $\tau_N > 0$ and that for all sub-coalitions be zero in our game (1). Because the owners now can only split $v(N)$-τ_N) and $mgp = mnbp$ remains unchanged, monopoly formation now requires:

$$v(N) - \tau_N \geq mnbp. \tag{13}$$

Thus, $\tau_N \leq (v(N) - mnbp)$ ($\tau_N > [v(N) - mnbp]$) holds for a successful (failed) monopoly formation.

In other words, the difference between $v(N)$ and $mnbp$ serves as an upper (lower) bound of monopoly's transaction costs below (above) which monopoly formation is possible (impossible), which can be empirically estimated.

5. THE NTU COASE THEOREM

This section studies the efficient payoffs and efficient firms in an NTU coalitional economy. An NTU coalitional economy (also called NTU bargaining economy, NTU coalitional game, and NTU game in characteristic form) is defined as

$$\Gamma = \{N, V(\cdot)\}, \tag{14}$$

which specifies, for each $S \subseteq N$, a non-empty set of payoffs $V(S)$ in $\mathbf{R}^S$, which is the Euclidean space whose dimension and coordinates are the number of players in S and their payoffs. Let the (weakly) efficient set of each $V(S)$ be denoted by

$$\partial V(S) = \{\ y \in V(S) |\ \text{there is no}\ x \in V(S)\ \text{such that}\ x >> y\},$$

where vector inequalities are defined by: $x \geq y \Leftrightarrow x_i \geq y_i$, all i; $x > y \Leftrightarrow x \geq y$ and $x \neq y$; and $x >> y \Leftrightarrow x_i > y_i$, all i.

Scarf (1967b) introduced the following two assumptions for (14): *i*) each $V(S)$ is closed and comprehensive (i.e., $y \in V(S), u \in R^S$ and $u \leq y$ imply $u \in V(S)$); *ii*) for each S, $\{y \in V(S) | y_i \geq \partial V(i)$, all $i \in S\}$ is non-empty and bounded, where $\partial V(i) = \max\{x_i | x_i \in V(i)\} > 0$. Under these two assumptions, each $\partial V(S)$ is closed, non-empty and bounded.

Given $S \subseteq N$, a payoff vector $u \in R^n$ is blocked by S if S can obtain a higher payoff for each of its members than that given by u, or precisely if there is $y \in V(S)$ such that $y_S >> u_S = \{u_i | i \in S\}$ or $u_S \in V(S) \backslash \partial V(S)$. A payoff vector $u \in \partial V(N)$ is in the usual core if it is unblocked by all $S \neq N$, so the usual core of (14) can be given as

$$C_0(\Gamma) = \{u \in \partial V(N) | u_S \notin V(S) \backslash \partial V(S),\ \text{all}\ S \neq N\}. \tag{15}$$

Balanced NTU games can be defined geometrically as below. For each $S \neq N$, let $\widetilde{V}(S) = V(S) \times R^{-S} \subset R^n$, where $R^{-S} = \Pi_{i \notin S} R^i$. For each minimal balanced collection $B \in \Phi$ in (5), let

$$GP(B) = \cap_{S \in B} \widetilde{V}(S),\ \text{and}\ GP(\Gamma) = \cup_{B \in \Phi} GP(B) \tag{16}$$

denote the payoffs generated by B and the set of generated-payoffs. Note that $GP(B)$ is simplified to $GP(B) = \Pi_{S \in B} V(S)$ when B is a partition. Similar to the TU case, we only need to consider minimal balanced collections because non-minimal balanced collections don't generate additional payoffs. Now, we are ready to define the efficient generated-payoffs $EGP = EGP(\Gamma) = \partial GP(\Gamma)$, and the efficient payoffs $EP = EP(\Gamma) = \partial(GP(\Gamma) \cup V(N))$, which are the NTU counterparts of *mgp* and *mp* in (6-7).

Definition 5.1 *Given game* (14) *and its GP in* (16)*, its EGP and EP are given by*

$$EGP = \{y \in GP(\Gamma) | \exists\ no\ x \in GP(\Gamma)\ such\ that\ x >> y\},\ and \tag{17}$$

$$EP = \{y \in GP(\Gamma) \cup V(N) | \exists\ no\ x \in GP(\Gamma) \cup V(N)\ with\ x >> y\}. \tag{18}$$

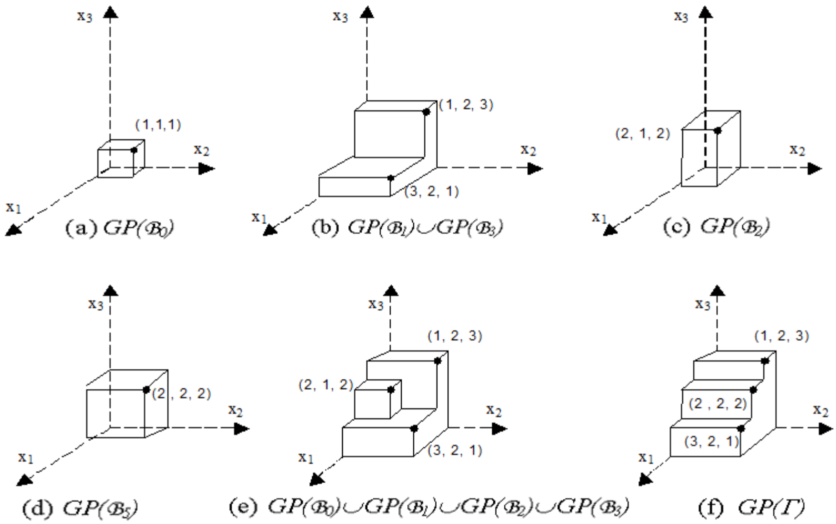

Figure 1: The generated payoffs in Example 5.1, where $B_0 = \{1,2,3\}$, $B_1 = \{1,23\}$, $B_2 = \{2,13\}$, $B_3 = \{3,12\}$ and $B_5 = \{12,13,23\}$.

Readers are encouraged to visualize the generated-payoffs in the following example, which are illustrated in Figure 1.

Example 5.1 *$n = 3$, $V(i) = \{x_i | x_i \leq 1\}$, $i = 1,2,3$; $V(12) = \{(x_1,x_2)|(x_1,x_2) \leq (3,2)\}$, $V(13) = \{(x_1,x_3)|(x_1,x_3) \leq (2,2)\}$, $V(23) = \{(x_2,x_3)|(x_2,x_3) \leq (2,3)\}$, $V(123) = \{x | x_1 + x_2 + x_3 \leq 5\}$. Let B_i, $i = 0,1,2,3$, be as in Example 2.1, and $B_5 = \{12,13,23\}$. Then, $GP\ (B_0) = \{x|x \leq (1,1,1)\}$, $GPB_1) = \{x|x \leq (1,2,3)\}$, $GPB_2) = \{x|x \leq (2,1,2)\}$, $GPB_3) = \{x|x \leq (3,2,1)\}$, and $GPB_5) = \{x|x \leq (2,2,2)\}$.*

Now, the game (14) is balanced if $GP(\Gamma) \subset V(N)$ or if for each balanced B, $u \in V(N)$ must hold if $u_S \in V(S)$ for all $S \in B$. To see a balanced game geometrically, visualize that one is flying over a city. Treat the generated-payoffs $GP(\Gamma)$ as trees and buildings in the city and the grand coalition's payoff $V(N)$ as clouds. Then, a game is balanced if one sees only clouds and unbalanced if one sees at least one building or tree top above the clouds. In Figure 2b for Example 5.1, one sees three building tops above the clouds so the game is unbalanced; in Figure 2a for the following Example 5.2, one sees only clouds so the game is now balanced.

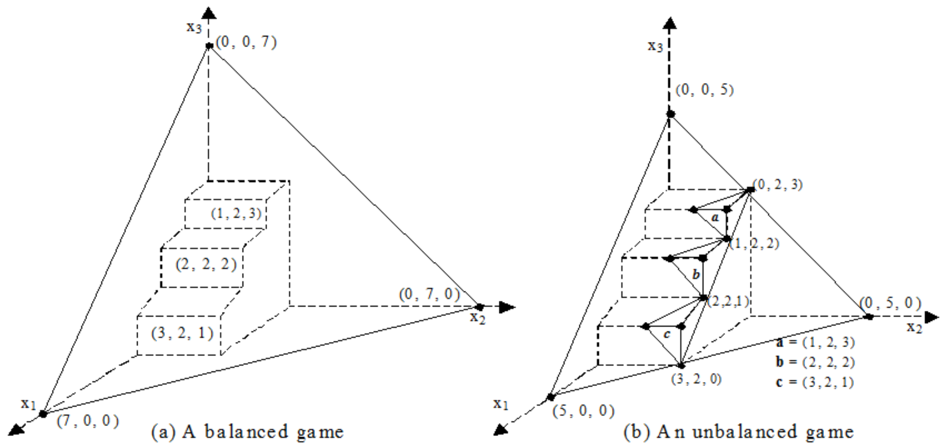

Figure 2: Balanced and unbalanced games.

Example 5.2 *Same as Example 5.1, except* $V(123) = \{x | x_1 + x_2 + x_3 \leq 7\}$.

Note that the collection $B_5 = \{12, 13, 23\}$ in Example 5.1 generates new payoffs that are outside of those generated by the four partitions and are better than $V(N)$ as shown by the difference between (e) and (f) in Figure 1. Needless to say, it is the discovery of such new and better payoffs that gives rise to our mathematical versions of the Coase theorem.

Recall that a payoff vector u is unblocked by S if $u \in [V(S) \backslash \partial V(S)]^C \times R^{-S}$ or if $u_S \notin V(S) \backslash \partial V(S)$, where the superscript C denotes the complement of a set. The following concept of minimum no-blocking frontier is the NTU counterpart of *mnbp* in (8):

Definition 5.2 *Given game (14), the set of payoffs unblocked by all* $S \neq N$ $(UBP = UBP(\Gamma))$ *and the minimum no-blocking frontier* $(MNBF = MNBF(\Gamma) = \partial UBP)$ *are respectively given by*

$$UBP = \cap_{S \neq N} \{[V(S) \backslash \partial V(S)]^C \times R^{-S}\} \subset R^n, \text{ and} \tag{19}$$

$$MNBF = \{y \in UBP(\Gamma) | \exists \text{ no } x \in UBP(\Gamma) \text{ with } x << y\}. \tag{20}$$

It is easy to see that each payoff vector on or above $MNBF(\Gamma)$ is unblocked by all $S \neq N$, and the usual core can be given as $C_0(\Gamma) = UBP(\Gamma) \cap \partial V(N) = MNBF(\Gamma) \cap \partial V(N)$. Similar to the TU case, *MNBF* represents sub-coalitions' power to block the grand coalition's proposals. Theorem 3 below shows that sub-coalitions' blocking power and producing ability are also dual to each other in coalitional NTU games, which is the NTU counterpart of Theorem 1.

Theorem 5.1 *Given game (14), its minimum no-blocking frontier and efficient generated-payoffs have a non-empty intersection.*

To put it differently, the NTU counterpart of *mnbp* = *mgp* in game (1) is that the set of unblocked and efficient generated-payoffs is non-empty, or precisely,

$$Z(\Gamma) = MNBF(\Gamma) \cap EGP(\Gamma) \neq \varnothing. \tag{21}$$

It is straightforward to verify $a, b, c \in Z(\Gamma)$ in Example 5.1 (see Figure 2b), where $a = (1,2,3)$, $b = (2,2,2)$, and $c = (3,2,1)$, so $Z(\Gamma) \neq \varnothing$ holds in the example.

Recall that $EGP(\Gamma) \subseteq V(N)$ holds in balanced games. Then, $MNBF(\Gamma) \cap EGP(\Gamma) \neq \varnothing$ implies $MNBF(\Gamma) \cap \partial V(N) = C_0(\Gamma) \neq \varnothing$in balanced games. Hence, the above duality implies Scarf's core theorem Scarf (1967b). It also implies two new existence theorems on the usual core, which are summarized in the following corollary:

Corollary 5.2 *Given game (14), the following three claims hold: i) its usual core is non-empty if it is balanced (Scarf, 1967b); ii) its usual core is non-empty if and only if the grand coalition's payoff is large enough to guarantee no blocking; and iii) its usual core is non-empty if players can't produce better payoffs than the grand coalition's payoff.*

The above results in parts *i*) and *iii*) can be precisely stated in one argument: $C_0(\Gamma) \neq \varnothing$ if $GP(\Gamma) \subset V(N)$, and the result in part *ii*) can be precisely stated as: $C_0(\Gamma) \neq \varnothing \Leftrightarrow$ there exists $x \in \partial V(N)$and $y \in MNBF(\Gamma)$ such that $x \geq y$.

Due to the generality of non-transferable utilities, the above NTU core results are more general than the earlier TU results in at least three aspects: *i*) the usual NTU core is no longer convex, as shown in Figure 3 for Examples 5.1-5.2; *ii*) balancedness is only sufficient and no longer necessary for a non-empty usual NTU core, see Figure 3a for a non-empty usual NTU core in the unbalanced Example 5.1; and *iii*) "players can't produce better payoffs than $V(N)$" is no longer necessary for a non-empty usual NTU core.

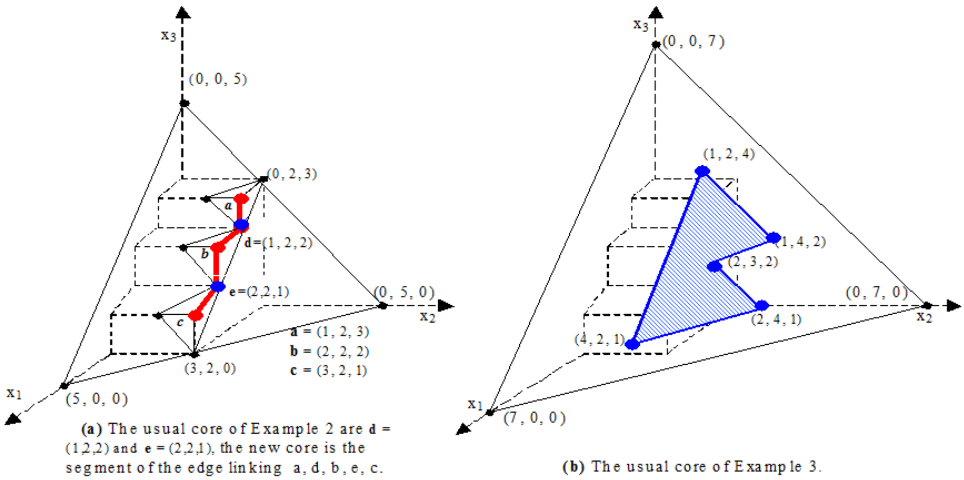

Figure 3: The usual core and the new core: blue-colored payoffs are the usual core, and red-colored payoffs are the new core with the usual core removed.

Recall that $Z(\Gamma)$ in (21) is the set of unblocked and efficient generated-payoffs. Let the subset of $Z(\Gamma)$ that are also unblocked by the grand coalition N be denoted by

$$Z^*(\Gamma) = Z(\Gamma) \cap [V(N) \backslash \partial V(N)]^C. \tag{22}$$

Now, let the set of minimal balanced collections that generate $Z(\Gamma)$ and $Z^*(\Gamma)$ be denoted respectively by

$$\begin{aligned} D_0(\Gamma) &= \{B \in \Phi | GP(B) \in Z(\Gamma)\}, \text{ and} &(23)\\ D_1(\Gamma) &= \{B \in D_0(\Gamma) | GP(B) \in Z^*(\Gamma)\}. &(24)\end{aligned}$$

Note that $D_0(\Gamma)$ is the NTU counterpart of the minimal set $B_0(\Gamma)$ in (11). Now, by replacing the grand coalition's payoff $V(N)$ with the efficient payoffs $EP(\Gamma)$ in (18), one gets the following new NTU core (or simply the NTU core) $C(\Gamma)$ and the set of efficient firms $D^*(\Gamma)$:

$$C(\Gamma) = \{u \in EP(\Gamma) | u_S \notin V(S) \backslash \partial V(S), \text{ all } S \subseteq N\} \tag{25}$$

$$= \begin{cases} C_0 \text{ if } GP \subseteq V(N) \\ Z \text{ if } V(N) \subseteq GP \backslash \partial GP \text{ or if } V(N) \subsetneq GP \backslash \partial GP, GP \subsetneq V(N), C_0 = \varnothing \\ C_0 \cup Z^* \text{ if } V(N) \subsetneq GP \backslash \partial GP, GP \subsetneq V(N), C_0 \neq \varnothing, \end{cases}$$

$$D^*(\Gamma) = \text{The set of efficient firms or set of efficient collections} \tag{26}$$
$$= \begin{cases} \{N\} \text{ if } GP \subseteq V(N) \\ D_0 \text{ if } V(N) \subseteq GP \backslash \partial GP \text{ or if } V(N) \subsetneq GP \backslash \partial GP, GP \subsetneq V(N), C_0 = \varnothing \\ \{N\} \cup D_1 \text{ if } V(N) \subsetneq GP \backslash \partial GP, GP \subsetneq V(N), C_0 \neq \varnothing, \end{cases}$$

where $C_0 = C_0(\Gamma)$, $GP = GP(\Gamma)$, $Z = Z(\Gamma)$, $Z^* = Z^*(\Gamma)$, $D_0 = D_0(\Gamma)$ and $D_I = D_I(\Gamma)$ are, respectively, the usual core in (15), generated payoffs in (16), unblocked and efficient generated-payoffs in (21), efficient generated-payoffs that are also unblocked by grand coalition in (22), collections supporting $Z(\Gamma)$in (23), and collections supporting $Z^*(\Gamma)$ in (24). In words, the NTU core is characterized in three cases and it is equal to:

i) the usual NTU core if the game is balanced;

ii) the set of unblocked and efficient generated-payoffs if players can produce better payoffs than $V(N)$ or if players cannot produce better payoffs than $V(N)$ and the game is unbalanced with an empty usual core; and

iii) the union of the usual NTU core and a subset of unblocked and efficient generated-payoffs (i.e., $C_0 \cup Z^*$) if players cannot produce better payoffs than $V(N)$ and the game is unbalanced with a non-empty usual NTU core.

Now, our NTU Coase theorem, comprising the above answers, can be stated as:

Theorem 5.3 *In coalitional NTU economy* (14) *without transaction costs, owners will produce the efficient payoffs by allocating inputs to efficient firms based on the firms' shadow values, and will choose an efficient payoff vector from the non-empty NTU core.*

Precisely, the efficient payoffs, core payoffs and efficient firms are given in (18), (25) and (26), respectively. Figure 3a illustrates the difference between the usual NTU core (i.e., points *d* and *e*) and new NTU core (i.e., the segment of the edge linking all three peaks) in Example 5.1. Note that efficient firms in (26) are defined according to the three cases of the core in (25). There are five sets of efficient firms in Example 5.1: $B_1 = \{23, 1\}$, $B_2 = \{13, 2\}$, $B_3 = \{12, 3\}$, $B_m = \{N\}$ and $B_5 = \{12, 13, 23\}$. Keep in mind that efficiency here is defined by weakly efficient solutions. For example, the payoff $(2,1,2)$ is only weakly efficient; it is not efficient in the sense of Pareto because it is Pareto-dominated by $(2,2,2)$.

Analogous to the TU case, the strong conclusion of NTU Coase theorem results from the advantages of utilizing generated-payoffs: in the usual NTU core, players just choose from $\partial V(N)$; whereas in the new NTU core, players choose from the game's efficient payoffs, which are sometimes better than $\partial V(N)$.

6. CONCLUSION AND DISCUSSION

The above analysis has explored the possibility that owners in a coalitional economy sometimes could produce better payoffs than the monopoly payoff. It has revealed that a firm's shadow value is equal to its balancing weight, and it has captured the major aspects and main insights of the Coase theorem by establishing two mathematical theorems in coalitional economies.

By modeling non-market allocation of resources as a coalitional economy or bargaining economy, the paper not only has advanced coalition formation from partitions to minimal balanced collections but also has advanced the study of the Coase theorem in three areas. First, our two versions of the Coase theorem (i.e., Theorems 2 and 4) show that it is sometimes socially optimal for firms to shut down parts of their operations and for workers to have two or more part-time jobs. This conclusion, previously unreported, will provide a new line of argument in studying labor theory, production theory, and other related fields in economics.

Second, our two versions show precisely how the size of transaction costs in each merger or coalition could prevent or allow its formation, and this provides a two-step procedure for empirically estimating the size of transaction costs involved in each previous or future application of the Coase theorem: i) identify the merger (or the parties in the transaction problem under investigation) and convert it into a coalitional economy, and ii) compute its minimum no-blocking payoff ($mnbp$). The difference between the merger's payoff and its $mnbp$ is the estimated upper (lower) bound of transaction costs below (above) which the optimal outcome predicted by the Coase theorem holds (fails).

Finally, our two versions have the potential to open doors for applying the Coase theorem not only to TU and NTU transaction problems but also to all non-transaction problems that are modeled by coalitional games.

Appendix

Proof of Theorem 1: For each $S \neq N$, let $e_S = (x_1, \ldots, x_n)' \in \mathbf{R}^n_+$ be its incidence vector or the column vector such that $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \notin S$, and $e = e_N = (1, \ldots, 1)'$ be a column vector of ones. Then, the dual problem for the minimization problem (8) is the following maximization problem:

$$Max\{\Sigma_{S\neq N} y_S v(S) | y_S \geq 0, \text{ all } S \neq N; \text{ and } \Sigma_{S\neq N} y_S e_S \leq e\}. \tag{27}$$

We will show that (27) is equivalent to the maximization problem (6). First, we show that the inequality constraints in (27) can be replaced by equation constraints.

Let $Ay \leq e$ and $y \geq 0$ denote the constraints in (27), where $A = A_{n\times(2^n-2)} = [e_S | S \neq N]$ is the constraint matrix, and y is the $(2^n - 2)$ dimensional vector whose indices are the proper coalitions. Let the rows of A be $a_1, \ldots, a_n$, and for each feasible y, let $T = T(y) = \{i | a_i y < 1\}$ be the set of loose constraints, so $N \backslash T = \{i | a_i y = 1\}$ is the set of binding constraints.

If $T(y) \neq \varnothing$, let z be defined as: $z_S = y_S + (1 - a_i y)$ if $S = \{i\}$, for each $i \in T$, and $z_S = y_S$ if $S \neq \{i\}$ for all $i \in T$. One sees that $z > y$ and $T(z) = \varnothing$. Hence, for any y with $T(y) \neq \varnothing$, there exists $z \geq 0, Az = e$ such that $\Sigma_{S\neq N} y_S v(S) \leq \Sigma_{S\neq N} z_S v(S)$. Thus, the feasible set of (27) can be reduced to $\{z | z \geq 0, Az = e\}$, without affecting the maximum value. So the maximization problem in (27) is equivalent to the following problem:

$$Max\{\Sigma_{S\neq N} y_S v(S) | Ay = e \text{ and } y \geq 0\}. \tag{28}$$

Next, we establish the one-to-one relationship between the extreme points of (28) and the minimal balanced collections. Note that for each feasible y in (28), $B(y) = \{S | y_S > 0\}$ is a balanced collection. Let y be an extreme point of (28). We now show that $B(y) = \{S | y_S > 0\}$ is a minimal balanced collection.

Assume by way of contradiction that $B(y)$ is not minimal, then there exists a balanced subcollection $B \subset B(y)$ with balancing vector z. Note that $z_S > 0$ implies $y_S > 0$. Therefore, for a sufficiently small $t > 0$ (e.g., $0 < t \leq 1/2$, and $t \leq Min\{y_S / |z_S - y_S| |$ all S with $y_S \neq z_S\}$), one has

$$w = y - t(y - z) \geq 0, w' = y + t(y - z) \geq 0.$$

$Ay = e$ and $Az = e$ lead to $Aw = e$ and $Aw' = e$. But $y = (w + w')/2$ and $w \neq w'$ contradict the assumption that y is an extreme point. So $B(y)$ must be minimal.

Now, let $B = \{T_1, \ldots, T_k\}$ be a minimal balanced collection with a balancing vector z. We need to show that z is an extreme point of (28). Assume again by way of contradiction that z is not an extreme point, so there exist feasible $w \neq w\prime$ in (28) such that $z = (w+w')/2$. By $w \geq 0$, $w' \geq 0$, one has

$$\{S|w_S > 0\} \subseteq B = \{S|z_S > 0\}, \text{and } \{S|w'_S > 0\} \subseteq B = \{S|z_S > 0\}.$$

The above two expressions show that both w and w' are balancing vectors for some subcollections of B. Because B is minimal, one has $w = w' = z$, which contradicts $w \neq w'$. Therefore, z must be an extreme point of (28).

Finally, by the standard results in linear programming, the maximal value of (28) is achieved among the set of its extreme points, which are equivalent to the set of the minimal balanced collections, so (28) is equivalent to $Max\{\Sigma_{S\in B} y_S v(S)|B \in \Phi\}$, where $B \in \Phi$ in (5) is a minimal balanced collection with the balancing vector y. This shows that (27) is equivalent to the maximization problem (6) for mgp, which completes the proof. **Q.E.D**

Proof of Corollary 1: The equivalences follow from Theorem 1. Note that part ii) was established in previous studies using the duality theorem for $Min\{\Sigma_{i\in N} x_i | \Sigma_{i\in S} x_i \geq v(S), all\ S \subseteq N\}$. **Q.E.D**

Proof of Theorem 2: Discussions before the theorem serve as a proof. **Q.E.D**

Our proof for Theorem 3 uses the following lemma on open covering of the simplex $\Delta^N = X(1) = \{x \in \mathbf{R}^n_+ | \Sigma_{i\in N} x_i = 1\}$.

Lemma 1 (Scarf, 1967a; Zhou, 1994): *Let $\{C_S\}$, $S \neq N$, be a family of open subsets of Δ^N that satisfy $\Delta_{N\backslash\{i\}} = \{x \in \Delta^N | x_i = 0\} \subset C_{\{i\}}$ for all $i \in N$, and $\cup_{S\neq N} C_S = \Delta^N$, then there exists a balanced collection of coalitions B such that $\cap_{S\in B} C_S \neq \varnothing$.*

Proof of Theorem 3: Let UBP be the set of unblocked payoffs in (19), and EGP be the boundary or (weakly) efficient set of the generated payoff in (17). We shall first show that $UBP \cap EGP \neq \varnothing$.

For each coalition $S \neq N$, let $W_S = \{IntV(S) \times R^{-S}\} \cap EGP$ be an open (relatively in EGP) subset of EGP, where $IntV(S) = V(S) \backslash \partial V(S)$ is the interior of $V(S)$. For each minimal balanced collection of coalitions B, we claim that

$$\cap_{S\in B} W_S = \varnothing \tag{29}$$

holds. If (29) is false, there exists $y \in EGP$ and $y \in IntV(S) \times \mathbf{R}^{-S}$ for each $S \in B$. We can now find a small $t > 0$ such that $y + te \in IntV(S) \times \mathbf{R}^{-S}$ for each $S \in B$, where e is the vector of ones. By the definitions in (16)-(17), $y+te \in GP(B) = \cap_{S\in B}\{V(S)\times \mathbf{R}^{-S}\} \subset GP$, which contradicts $y \in EGP$. This proves (29).

Now, suppose by way of contradiction that $UBP \cap EGP = \varnothing$. Then, $EGP \subset UBP^C$, where superscript C denotes the complement of a set. The definition of W_S and

$$UBP^C = \{\cap_{S\neq N}\{[V(S)\backslash \partial V(S)]^C \times \mathbf{R}^{-S}\}\}^C = \cup_{S\neq N}\{IntV(S)\times \mathbf{R}^{-S}\}$$

together lead to $\cup_{S\neq N} W_S = EGP$, so $\{W_S\}, S \neq N$, is an open cover of EGP.

Because the set of generated payoffs is comprehensive and bounded from above, and the origin is in its interior (by $\partial V(i) > 0$, all i), the following mapping from EGP to Δ^N:

$$f : x \to x/\Sigma x_i,$$

is a homeomorphism. Define $C_S = f(W_S)$ for all $S \subseteq N$, one sees that $\{C_S\}, S \neq N$, is an open cover of $\Delta^N = f(EGP)$.

For each $i \in N$, $\partial V(i) > 0$ leads to $EGP \cap \{x \in \mathbf{R}^n | x_i = 0\} \subset W_{\{i\}}$, which in turn leads to $\Delta_{N\backslash\{i\}} = \{x \in \Delta^N | x_i = 0\} = f(EGP \cap \{x \in \mathbf{R}^n | x_i = 0\}) \subset C_{\{i\}} = f(W_{\{i\}})$. Therefore, $\{C_S\}$, $S \neq N$, is an open cover of Δ^N satisfying the conditions of Lemma 1, so there exists a balanced collection of coalitions B_0 such that $\cap_{S\in B_0} C_S \neq \varnothing$, which leads to $\cap_{S\in B_0} W_S \neq \varnothing$. This contradicts (29). Hence, $UBP \cap EGP \neq \varnothing$.

For each $x \in UBP \cap EGP$, we claim $x \in MNBF$. If this is false, we can find a small $\tau > 0$ such that $x - \tau e \in UBP$. Let $B \in \Phi$ be the minimal balanced collection of coalitions such that $x \in GP(B) = \cap_{S\in B}\{V(S)\times \mathbf{R}^{-S}\}$. Then, $x - \tau e \in IntV(S) \times \mathbf{R}^{-S}$ for each $S \in B$, which contradicts $x - \tau e \in UBP$. Therefore, $MNBF \cap EGP = UBP \cap EGP \neq \varnothing$. **Q.E.D**

Proofs of Corollary 2 and Theorem 4: The discussions preceding the corollary and the theorem serve as their proofs. **Q.E.D**

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