jMID

Journal of Mechanism and Institution Design

CONTENTS

Volume 4, Issue 1, November 2019
ISSN: 2399-844X (Print), 2399-8458 (Online)

Editorial board

Published by
The Society for the Promotion of Mechanism and Institution Design
Editorial office, Centre for Mechanism and Institution Design
University of York, Heslington, York YO10 5DD
United Kingdom
http://www.mechanism-design.org

ISSN: 2399-844X (Print), 2399-8458 (Online), DOI: 10.22574/jmid

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Printed in York and Klagenfurt by the Centre for Mechanism and Institution Design

A Letter from the Editor

For the current issue, I would like to report to our readers the following important items concerning the *Journal of Mechanism and Institution Design* and its *Society for the Promotion of Mechanism and Institution Design*.

Our Society as a registered UK charity body (no. 1174289), an independent learned society, will hold its second *Conference on Mechanism and Institution Design* at the Alpen-Adria-University of Klagenfurt, Klagenfurt, Austria, on Thursday-Saturday, 11th-13th June 2020. The confirmed keynote speakers are Professors Pierpaolo Battigalli (Bocconi University), Johannes Hörner (Yale University), and Benny Moldovanu (University of Bonn). The organiser is Professor Paul Schweinzer of the Alpen-Adria-University of Klagenfurt. Up to date conference information can be found at our Society website http://www.mechanism-design.org/news.php.

By the end of this year our colleagues on the editorial board Sayantan Ghosal, David Pérz-Castrillo, Neil Rankin, and Arunava Sen will complete their term. We wish to thank them for their excellent service, advice and support during the formative years of the Journal. At the same time, we warmly welcome four outstanding colleagues to join our editorial board from January 2020. They are Youngsub Chun of Seoul National University, Flip Klijn of Universitat Autònoma de Barcelona, Jorgen Kratz of University of York, and Takuro Yamashita of Toulouse School of Economics. We are looking forward to working with them towards further advancement of the Journal.

As our Journal will enter its fifth year in 2020 and the number of submissions is growing, we may start to gradually increase the issues of each volume. This, however, does not mean we will sacrifice the quality of published articles. We will always remain true to our guiding principle: To publish original and innovative high quality research works in the field of mechanism and institution design and to provide the best possible services to the profession for the public interest. The Journal is totally free of charge to the authors (no submission fee, no membership fee, and no publication fee) and freely open to everyone on the Internet. Our Journal, however, does rely on donations and has received generous financial support. We firmly believe that open access journals like ours are the future of scholarly publishing.

Your support – whether in the form of donation, submission or reviewing – will be always appreciated and play a vital role in keeping our vision alive and helping the Journal thrive.

Zaifu Yang, York, 29th October, 2019

Journal of Mechanism and Institution Design
ISSN: 2399-844X (Print), 2399-8458 (Online)
DOI:10.22574/jmid.2019.11.001

EFFICIENT AND DOMINANCE SOLVABLE AUCTIONS WITH INTERDEPENDENT VALUATIONS

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ABSTRACT

In auction environments in which agents have private values, the Vickrey auction induces agents to truthfully reveal their preferences and selects the efficient allocation accordingly. When the agents' valuations are interdependent, various generalizations of the Vickrey auction have been found which provide incentives for truthful revelation of all private information and preserve efficiency. However, these mechanisms generally do not provide the bidders with dominant strategies. The existing literature has therefore used a stronger equilibrium solution concept. In this paper we show that while the generalized VCG mechanism admits a multiplicity of equilibria, many of which are inefficient. We give conditions under which the efficiency equilibrium is the unique outcome of iterative elimination of *ex post* weakly dominated strategies. With two bidders, the standard single-crossing condition is sufficient. With more than two bidders, we show by example that a strengthening of the single-crossing condition is necessary.

Keywords: Generalized VCG mechanism, iterative elimination of ex post weakly dominated strategies.

JEL Classification Numbers: D44, D82.

This paper was first circulated in 2000. The literature has since grown much bigger than is reflected in our references. We apologize for not being able to do justice to this subsequent literature. We thank Dirk Bergemann, Eddie Dekel, Stephen Morris, and Asher Wolinsky for their continual interest in this old paper, and Zaifu Yang for inviting us to submit it to this *Journal*. All errors are ours.

1. INTRODUCTION

The Vickrey-Clarke-Groves mechanism is among the pillars of mechanism design and implementation theory. In its simplest form, the Vickrey auction is a selling mechanism under which bidders with private valuations for a single object have a dominant strategy to submit a bid equal to their valuation, thereby ensuring that the auction will be won by the bidder whose valuation is the highest. More generally, in social choice environments in which agents have private values (they have all the information that is relevant to determine their preferences), and utilities are quasi-linear in money, the Vickrey-Clarke-Groves (VCG) mechanism induces agents to truthfully reveal their preferences, and selects the efficient outcome accordingly.

Attention has recently turned to efficient implementation in environments characterized by *interdependent* values. Values are interdependent whenever the preferences of one agent depends on some information held privately by another agent. To achieve an *ex post* efficient allocation in such an environment, a planner must induce truthful revelation of all private information with the understanding that the social alternative will be chosen to maximize total surplus as calculated on the basis of this information. For a simple example of such an environment, consider a planner who must decide how to allocate a single indivisible object among a given set of agents. When the agents' valuations for the object are interdependent, various generalized versions of the Vickrey auction have been found to provide incentives for truthful revelation and efficiently allocate the object.[1]

However, unlike in private value contexts, the generalized VCG mechanism does not provide the players with dominant-strategy incentives for truthful revelation. Instead, in these interdependent valuation contexts, the implementation notion used has been weakened to an equilibrium concept. Specifically, it has been shown that the generalized VCG mechanism admits an *ex post* Nash equilibrium in which all agents report truthfully. While ex post equilibrium is a more conservative equilibrium concept than, for example, Bayesian Nash equilibrium, it nevertheless presumes considerable coordination on the part of the players. In particular, there is generally a vast multiplicity of ex post

[1] See Maskin (1992), Dasgupta & Maskin (2000), Perry & Reny (1999), Bergemann & Välimäki (2002), Ausubel (1999), Bergemann & Välimäki (2002), and Krishna (2003). Each of these papers, including the present one, assumes a single-dimensional type for each bidder. Jehiel & Moldovanu (2001) prove impossibility results when types are multi-dimensional.

equilibria, most of which are not efficient. Thus, while the existing literature has shown how the VCG mechanism can be extended to interdependent valuation settings, these generalizations have not been shown to have the same robustness as their private value counterparts.

In this paper we show some progress on this dimension. We study ex post efficient implementation in single object and multi-unit auction environments with interdependent valuations. We find conditions under which the efficient ex post equilibrium of a version of the generalized Vickrey auction is the unique outcome of iterative elimination of weakly dominated strategies. When there are two bidders and any number of identical objects, a sufficient condition for dominance solvability is a standard single-crossing condition. This condition has been observed in the literature as essentially necessary for efficient implementation even in ex post equilibrium. Thus, when there are two bidders, the generalized Vickrey auction is dominance solvable (essentially) whenever it has an efficient ex post equilibrium.

When there are more than two bidders, dominance solvability becomes a strictly stronger requirement than implementation in ex post equilibrium. We demonstrate this in the context of a symmetric linear model in which the necessary and sufficient conditions can be easily identified and interpreted. While the single crossing condition assumes only that the valuation of bidder i is more sensitive to the private information of bider i than that of other bidders, the necessary and sufficient condition we identify is that the influence of i's private type on his own valuation is greater than the influence of the *sum* of the types of the other bidders.

Interestingly however, even when our condition is not satisfied, our negative result for the case of 3 or more bidders does not conclusively rule out the existence of a dominance solvable efficient mechanism. This is because our condition applies to the class of generalized VCG mechanisms; and while it is known that all ex post efficient *direct revelation* mechanisms belong to this class, as we show in this paper, the revelation principle fails for our solution concept of iterative elimination of weakly dominated strategies. Thus, unlike nearly all implementation notions studied in the literature, it does not suffice for our solution concept to focus on direct revelation mechanisms. The possibility is therefore open that a dominance solvable, efficient *indirect* auction mechanism can be found when our condition fails.

Finally, it is well known that the outcome of iterative elimination of weakly dominated strategies often depends on the chosen order in which strategies are eliminated. We wish to show that the efficient ex post equilibrium is the outcome under any possible order of elimination. However, we argue by examples that in games such as auctions, order independent elimination is too much to ask for due simply to technical issues related to the infinity of strategies and payoff discontinuities. We therefore identify a weak restriction, which we call *vigilantness*, on the elimination procedures considered and prove that every order of elimination satisfying this restriction yields the efficient ex post equilibrium. In finite games, vigilantness reduces to the condition that the elimination procedure does not leave any dominated strategies in the ultimate solution. Thus, in finite versions of our auction environment, this latter condition is sufficient to ensure that the (finite) generalized VCG mechanism yields the efficient ex post equilibrium as the unique (i.e. order independent) dominance solution.

Dominance solvability of interdependent value auctions has been previously investigated by Harstad & Levin (1985). These authors study affiliated pure common value auctions with a special information structure: the highest among all bidders' signals is a sufficient statistic for the full profile of signals. They show that these auctions are interim dominance-solavle.

This paper is organized as follows. In section 2 we describe the class of environments we study, and we present the versions of weak dominance, and iterative elimination of weakly dominated strategies that we employ. The concept of vigilance is defined and discussed here. In section 3, we prove our result for the case of two bidders and a single object. In section 4, we show that using a (new) version of the generalized VCG mechanism, this result can be extended to the case of two bidders and any number of identical objects. In Section 5, we study the general case in which there are more than two bidders. In a model with symmetric, linear valuation functions, we identify necessary and sufficient conditions for the generalized VCG mechanism to be dominance solvable. Finally, in section 6 we present our counterexample to the revelation principle.

2. PRELIMINARIES

2.1. Dominance Solvability

For this subsection, we consider an arbitrary game of incomplete information. Each player $i \in \{1,\ldots,n\}$ has a set T_i of types, and a set A_i of actions. The payoff to player i is $\pi_i(a,s)$ when the type profile is $s \in T$ and the profile of chosen actions is $a \in A$. A pure strategy for player i is a map $\sigma_i : T_i \to A_i$ specifying an action for each possible type. Let Σ_i be the set of all pure strategies for i.

Definition 2.1. *Let $\hat{\Sigma}_{-i} \subset \Sigma_{-i}$ be a subset of strategy profiles for the opponents of i. Strategy $\sigma_i \in \Sigma_i$ ex post weakly dominates strategy $\hat{\sigma}_i$ against $\hat{\Sigma}_{-i}$ if for every type profile $s \in T$ and every $\sigma_{-i} \in \hat{\Sigma}_{-i}$,*

$$\pi_i(\sigma_i(s_i), \sigma_{-i}(s_{-i}), s) \geq \pi_i(\hat{\sigma}_i(s_i), \sigma_{-i}(s_{-i}), s)$$

with strict inequality for at least one $\sigma_{-i} \in \hat{\Sigma}_{-i}$ and s.

Ex-post weak dominance is the most conservative version of weak dominance possible in this incomplete-information setup. Other notions of dominance, such as ex-ante, or interim dominance, evaluate expected payoffs using some exogenous beliefs about the distribution of types. For any given belief, a strategy may be dominated in either of these senses without being ex-post dominated. On the other hand in finite games, if a strategy is ex-post dominated, then it is ex-ante and interim dominated for *any* full-support prior beliefs.[2]

In what follows, we will often say that an action a_i is dominated for a type s_i by another action $\hat{a}_i$ against a set of strategies. By this we mean that the payoff to s_i using $\hat{a}_i$ is at least as high as the payoff from using a_i and strictly higher in at least one possible case. Clearly a strategy σ_i is dominated if and only if $\sigma_i(s_i)$ is dominated for at least one s_i. If we say that a strategy σ_i is dominated within a given subset β of strategy profiles, we mean that $\sigma_i \in \beta_i$ and there is an element of β_i which dominates σ_i against β_{-i}. Finally, we say that a set β of strategy profiles is *internally undominated* if there is no strategy $\sigma_i \in \beta_i$ which is dominated within β.

In private value settings, the Vickrey auction implements the efficient allocation in dominant strategies. In general interdependent value settings, there is no efficient mechanism in which the players have a dominant strategy.

[2] The finite/full-support qualification is necessary here because otherwise an ex-post dominated strategy may satisfy the strict inequality in the definition only for a profile of types which have probability zero, in which case the strategy would not be dominated in the other senses. Of course our auction mechanisms have infinite strategy sets as an efficient auction in our setting must. It is nevertheless true in each of the dominance arguments we present that the strict inequality would be satisfied with positive probability under any full-support prior. We however do not take the time to prove this in each case below.

However, we will show below that efficiency can be achieved as the unique outcome of *iterative* elimination of ex-post weakly dominated strategies.

There are subtle aspects of iterative elimination of weakly dominated strategies, especially in games such as auctions in which there are infinitely many types and actions and in which payoffs are not continuous. In the remainder of this section, we formalize the types of elimination procedures we will consider.

In any game with finitely many strategy profiles Σ, an *elimination sequence* is a sequence β^k of strategy profiles satisfying the following four conditions

1. $\beta^0 = \Sigma$
2. $\beta^k \subset \beta^{k-1}$
3. $\sigma_i \in \beta_i^{k-1} \setminus \beta_i^k$ only if σ_i is dominated against β_{-i}^{k-1}
4. $\cap_k \beta^k$ is internally undominated.

It is well-known that even in finite games, the outcome of iterative elimination of weakly dominated strategies can depend on the chosen order in which strategies are eliminated. Our definition of an elimination sequence imposes no requirements on the order of elimination.

Our main focus is on environments with a continuum of types, in which an efficient mechanism must have infinitely many strategies.[3] In games with infinitely many strategies, the definition of an elimination sequence must be modified in two ways. First, we must strengthen the third requirement as follows.

3′. $\sigma_i \in \beta_i^{k-1} \setminus \beta_i^k$ only if σ_i is dominated against β_{-i}^{k-1} by an element of β_i^k.

That is, we add the requirement that eliminated strategies must be dominated by a strategy that survives. In games with infinite strategy sets and discontinuous payoffs, it is possible that each element of an infinite subset of strategies is dominated by another element of the same set, without any element of the set dominated by an undominated strategy. In our positive results below, when we

[3] Even if there are truly only finitely many types, the finite generalized VCG mechanism requires that the mechanism designer know precisely the set of possible types of each player. On the other hand, when there are two bidders, the second price auction can be used no matter what the size of the set of types. So if we are interested in mechanisms which minimize the informational demands on the designer, we are forced to consider infinite mechanisms.

construct an elimination sequence leading to the efficient equilibrium, we will be careful to show that all eliminated strategies are dominated by strategies that are not themselves dominated.

We wish also to show that the solutions we obtain by our elimination sequences would be obtained under any alternative order of elimination. However, when the mechanism has infinitely many strategies, the definition of an elimination sequence for finite games is far too permissive. We will modify[4] condition 4 as follows. Let $\mathscr{D}^k$ be the set of strategies that are dominated within β^{k-1} by an element of β^k.

4′. There exists a K such that if $\sigma_i \in \mathscr{D}^k$ for K consecutive rounds $k = k_1, \dots k_K$, then $\sigma_i \notin \beta^{k_K+1}$.

In finite games, condition 4′, which we call *vigilance*, is equivalent to 4. In infinite games, it is possible to construct pathological elimination sequences in which a strategy is dominated in every round of the process, but if not eliminated in any finite stage, is undominated within the final set of strategy profiles. Vigilance rules out such elimination sequences. Another possibility is that a set of strategies X can be dominated in every round, and if they are eliminated at any finite stage then a further set of strategies Y could be eliminated. With an invigilant elimination sequence it is possible that the set X is eliminated only “at the limit,” at which point it is no longer possible to eliminate Y.[5] Vigilance will ensure that all strategies in X are eliminated

[4] Condition 4′ is not a strict strengthening of condition 4. In some games, there are elimination sequences which satisfy 1, 2, 3′, and 4′, but which do not satisfy 4. In particular, it can happen that a strategy which is undominated in every round becomes dominated only in the limit set. The appropriate solution to such situations would be to extend the elimination procedure to trans-finite rounds. In our application, 4′ does imply 4 as this problem does not arise. We therefore do not introduce the necessary additional notation.

[5] Here is a two-player example illustrating both possibilities. Each player’s strategies are the integers. The payoffs are as follows. If $\sigma_i > 0$, then $\pi_i(\sigma_i, \sigma_{-i}) = 1 - 1/\sigma_i$ if $\sigma_i \leq \sigma_{-i} + 1$ and -1 otherwise. $\pi_i(0, \cdot) \equiv \varepsilon > 0$. For player 2, $\pi_2(\cdot, \sigma_2) \equiv -1$ for $\sigma_2 < 0$, and for player 1, when $\sigma_1 < 0$, $\pi_1(\sigma_1, \sigma_2) = 1$ if $\sigma_2 < 0$, otherwise ε if $\sigma_2 = 0$ and -1 if $\sigma_2 > 0$. Consider first the elimination sequence given by $\beta_1^k = \{\sigma_1 : \sigma_1 \leq 0 \text{ or } \sigma_1 > k\}$ and $\beta_2^k = \{0\} \cup \{\sigma_2 : \sigma_2 > k\}$. This elimination sequence is not vigilant because the negative strategies of player 1 are dominated in every stage after round 1, but are never eliminated. Note that the negative strategies are not dominated in the limit strategy set because none of the positive strategies remain despite the fact that for every finite k, there are infinitely many positive strategies in β^k. Hence condition 4 is satisfied in this case. Next consider the elimination sequence given by $\beta_1^k = \{\sigma_1 : \sigma_1 \leq 0 \text{ or } \sigma_1 > k\}$ and $\beta_2^k = \{0\} \cup \{\sigma_2 : |\sigma_2| > k\}$. This elimination sequence

in finite time. In infinite games, we define an elimination sequence to be a sequence satisfying conditions 1, 2, 3′, 4′.

Definition 2.2. *A dominance solution of an incomplete information game with strategy profiles Σ, is a subset $\beta \subset \Sigma$ such that for some elimination sequence β^k, $\beta = \cap_k \beta^k$.*

We pause here to note the role that vigilance plays in our results. We present two types of results below. First, for each of our mechanisms we exhibit an efficient dominance solution. Second, we argue that every dominance solution is efficient (recall that with weak dominance, we cannot take this for granted as we could with strict dominance which is order-independent). By limiting (albeit by what we argue is a purely technical restriction) the set of elimination sequences under consideration, we strengthen the first type of result, but weaken the second.

Note that for any game, eliminating all dominated strategies in every round yields an vigilant elimination sequence and hence every game has at least one dominance solution (although it may be empty).

Finally, while we focus on the solution concept of iterative elimination of weakly dominated strategies, we now recall the definition of ex post equilibrium, the solution concept previously studied in this context.

Definition 2.3. *An ex post equilibrium is a profile $(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_n)$ of strategy subsets with $\hat{\Sigma}_i \subset \Sigma_i$, such that for each strategy profile $(\sigma_1, \ldots, \sigma_n)$ with $\sigma_i \in \hat{\Sigma}_i$, each type profile s, and each player i,*

$$\pi_i(\sigma_i(s_i), \sigma_{-i}(s_{-i}), s) \geq \pi_i(a_i, \sigma_{-i}(s_{-i}), s)$$

for each action $a_i \in A_i$.

Our definition is non-standard because it ascribes sets of (pure) strategies to each player, rather than a unique strategy. Because the definition implies that each player i is (ex post) indifferent among each of the strategies in $\hat{\Sigma}_i$, regardless of the strategies in $\hat{\Sigma}_{-i}$ used by the remaining players, we can interpret this as a mixed-strategy ex post equilibrium. There is no need to specify

is not vigilant because all negative strategies for player 2 are dominated in every round, but it takes infinitely many rounds to eliminate them. If they were eliminated in finite time, as required by vigilance, then the negative strategies of player 1 could also be eliminated. Again, condition 4 is satisfied.

the randomization used by the players in this mixed-strategy equilibrium, because each player would be indifferent among all of his equilibrium strategies regardless of the mixtures used by the opponents.

2.2. Auctions with Interdependent Valuations

We consider auction settings with n bidders competing for a single object (we will generalize this framework to multi-unit auctions in Section 4). The set of bidders is $\mathscr{I}$. Each bidder i is assumed to observe a private type $s_i \in T_i = [0,1]$. Bidder i's value for the object depends on the realized profile of types $s \in T := \prod T_i$ according to the continuous function $v_i : T \to \mathbf{R}_+$, where v_i is strictly increasing and satisfies the following standard single-crossing condition:

Assumption 1 (Single-Crossing Property)**.** *For any $i \neq j$ and $\hat{s}_{-i}$, the difference*

$$g(s_i) = v_i(s_i, \hat{s}_{-i}) - v_j(s_i, \hat{s}_{-i})$$

crosses zero at most once, and from below.

When bidder i is awarded the object and makes payment t_i, his net payoff is $v_i(s) - t_i$.

We will consider direct and indirect auction mechanisms. An auction mechanism is a triple $(A, p, t) := (\{A_i\}_{i=1}^n, \{p_i\}_{i=1}^n, \{t_i\}_{i=1}^n)$. Here A_i is a set of actions available to bidder i. In a direct mechanism, these will be reports of bidders' private types, while in an indirect mechanism, they could be bids, or more complicated messages. For each i, $p_i : A \to [0,1]$ is a mapping specifying the probability $p_i(a)$ with which i is awarded the object when the profile of actions is a. The vector $p(a) = (p_1(a), \ldots, p_n(a))$ is called the outcome. Finally $t_i : A_i \to \mathbf{R}$ specifies the transfer made by bidder i to the auctioneer as a function of the chosen actions.

Any dominance solution β of an auction mechanism yields an outcome correspondence $f : T \rightrightarrows \Delta\{1, \ldots, n\}$. The set $f(s)$ of possible outcomes for type profile s is the set $\{p(a) : a = \sigma(s) \text{ for some } \sigma \in \beta\}$. For any $p \in \Delta\{1, \ldots, n\}$, let $C(p)$ denote the support of p. The (ex post) efficient allocation correspondence is e, defined by $p \in e(s)$ iff $C(p) \subset \operatorname{argmax}_i v_i(s)$. (We are implicitly assuming that it is never efficient for the auctioneer to keep the object.) We will say that a dominance solution β of an auction mechanism is efficient if its outcome correspondence satisfies $\emptyset \neq f(s) \subset e(s)$ for every s.

Definition 2.4. *The efficient allocation correspondence is ex post dominance implementable if there exists a mechanism of which every dominance solution is an efficient ex post equilibrium.*

Note that our implementation notion requires that the dominance solution also be an ex post equilibrium. Another potentially problematic feature of iterative elimination of dominated strategies in infinite games is that the dominance solution in general need not be a Nash equilibrium. In such a case, we would obviously lose faith in our proposed dominance solution, and this motivates the additional requirement. Fortunately, as we show, the generalized VCG mechanism does not suffer from this problem.

3. AUCTIONS WITH TWO BIDDERS

This section considers the case of two bidders and one object. We shall prove that the single crossing property is sufficient for efficient allocation to be ex post dominance implementable.

Theorem 3.1. *When there are two bidders, the efficient allocation is ex post dominance implementable.*

3.1. Illustration of the Proof

The proof of Theorem 3.1 is lengthy primarily because of the complications introduced by asymmetric valuation functions. However, the main idea can be understood with a simple diagram in the special case in which $v_1(0,0) = v_2(0,0)$ and $v_1(1,1) = v_2(1,1)$, i.e. symmetry at the extremes. We normalize these extreme values to 0 and 1 respectively.

In figure 1 we have depicted a box representing the set of type profiles in a two-bidder auction environment. The horizontal and vertical axes are s_1 and s_2 respectively. Our assumptions of continuity, monotonicity and single crossing imply that there is a continuous curve, which we will refer to as the "pivot curve" consisting of all type profiles in which the bidders' valuations are equal. In the special case we analyze here, the pivot curve connects the bottom left corner with the upper right corner of the box. By the single crossing condition, everywhere to the right of this curve, efficiency demands that the object be awarded to bidder 1, and to the left, bidder 2.

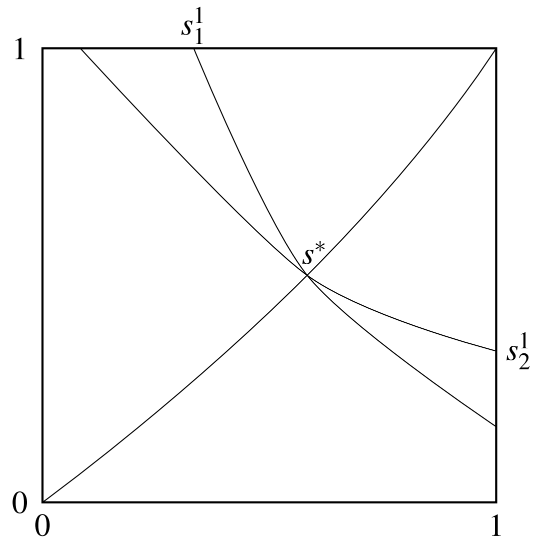

Figure 1: Pivot Curve and Indifference Curves

For every point s on the pivot curve, let $v(s)$ denote the bidders' common valuation for the object at type profile s. By monotonicity, the value $v(s)$ increases as s moves up the pivot curve. In a standard second-price auction, the following strategies constitute an ex post equilibrium (see Maskin (1992)). When bidder i has type s_i, he bids the value $v(s_i, s_{-i})$, where (s_i, s_{-i}) is the point on the pivot curve whose ith coordinate is s_i. That is, bidder i bids his *pivotal valuation*, which we denote by $b_i(s_i)$. Since this bidding behavior implies that 1 outbids 2 if and only if s is to the right of the pivot curve, this equilibrium yields an ex-post efficient allocation.

Unfortunately, this efficient equilibrium is not the unique ex post equilibrium of the second-price auction. In fact, there are uncountably many inefficient ex post equilibria. A simple class of inefficient ex post equilibria is constructed as follows.[6] Fix any $0 < \Delta \leq 1/2$. Modify the efficient bidding strategies so that each type s_1 such that $b_1(s_1) \in [1/2 - \Delta, 1/2 + \Delta]$ bids

[6] The following inefficient ex post equilibria translate naturally to inefficient undominated ex post equilibria of the English auction. Krishna (2003) provides examples of inefficient perfect Bayesian equilibria of the English auction, but his examples are either dominated or fail the "ex post regret" criterion. Applying our iterative dominance arguments to ascending auctions with interdependent valuations is left for future work.

$v_2(s_1, b_2^{-1}(1/2+\Delta)$, and each type s_2 such that $b_2(s_2) \in [1/2-\Delta, 1/2+\Delta]$ bids $v_1(b_1^{-1}(1/2-\Delta, s_2)$. Other types bid as in the efficient equilibrium. It can easily be checked that this is an ex post equilibrium, and the object is misallocated whenever $1/2-\Delta < b_1(s_1) < b_2(s_2) < 1/2+\Delta$.

This multiplicity may not be surprising at first; indeed even in the case of private values, there are inefficient equilibria of the second-price auction. However, with private values, all inefficient equilibria involve dominated strategies. This is not the case in our setting. When Δ is chosen sufficiently small, the strategies described above are not dominated. Furthermore, for any k, Δ can be chosen small enough that the resulting strategies are not dominated even after k rounds of elimination.

We now argue, however, that for any type s_i, any bid not equal to $b_i(s_i)$ would be eliminated after sufficiently many rounds of elimination. The first step is to eliminate for all types of either bidder bids greater than 1 and less than 0. Bids greater than 1 are dominated because the value of the object is always less than or equal to 1, and bids below 0 are dominated because the value of the object is always greater than or equal to 0.

Next consider a point s^* on the pivot curve. Let $b = b_1(s_1^*) = b_2(s_2^*)$ be the value that each bidder assigns to the object when the type profile is s^*. We can find "indifference curves" for each bidder i through the point s^*, connecting all of the type profiles at which bidder i assigns value b to the object. The single crossing condition implies that bidder 1's indifference curve lies above that of bidder 2 to the left of the pivot curve, and below to the right, as shown in figure 1.

Consider the type s_1^1 of bidder 1 for whom $v_1(s_1^1, 1) = b$. By monotonicity, for any type $s_1 < s_1^1$, the maximum possible value for the object, $v_1(s_1, 1)$ is strictly less than b. Moreover, for any such s_1, there is a bid $b' < b$ such that type s_1's valuation is less than or equal to b' for any possible type of bidder 2. By bidding b, type s_1 will win the auction and earn a strictly negative net payoff whenever the opponent bids between b' and b. By instead bidding b', s_1 will lose the auction in all these cases and obtain a payoff of zero. Since only in these cases does b' change the outcome relative to b, it follows that b' dominates b for type s_1.[7] Thus, we can eliminate a bid of b for any type of bidder 1 below s_1^1. By the same argument, we can eliminate b for any type of bidder 2 below s_2^1, and we can apply the analogous arguments for all possible

[7] This sketch ignores the possibility that b' itself may be eliminated. In fact, as we show in the proof, an undominated bid can be found which dominates both b' and b in this case.

bids corresponding to pivotal valuations.

Now consider type s_2^2, depicted in figure 2. This type is defined by $v_2(s_1^1, s_2^2) = b$. After having eliminated b for all types of bidder 1 less than s_1^1, bidder 2 can deduce, whenever he wins at a price b, that bidder 1's type is no less than s_1^1. This means in particular that conditional on winning at price b, the value of the object to type s_2^2 is at least b. And by monotonicity, for any type $s_2 > s_2^2$, the object would be worth strictly more than b.

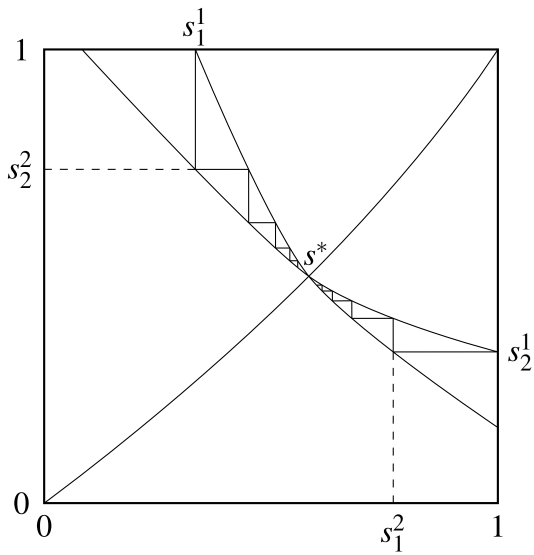

Figure 2: Iterative Elimination for b

We wish to conclude from this that b is dominated for such a type s_2 by some $b' > b$. To make this argument we must calculate the payoff to s_2 conditional on winning at prices slightly higher than b. The assumed continuity of the valuation functions implies that for bids near b, the set of types of bidder 1 eliminated in round one is not much different than the set eliminated for b, namely $[0, s_1^1)$. In particular, the smallest type that remains for a bid slightly higher than b is close to s_1^1. This means that we can find a b' higher than, but close enough to b such that conditional on winning against any bid between b and b' type s_2 is ensured of a strictly positive net payoff. It follows that b', which leads to a different allocation only in these cases, dominates b for s_2.[8]

[8] Again, a more careful argument is required to show that b is dominated by an undominated b'.

This argument shows that in the second round of elimination, b can be eliminated for every type of bidder 2 greater than s_2^2. The analogous argument eliminates b for all types of bidder 1 greater than s_1^2. As suggested by figure 2, we can now iterate this argument deriving bounds s_1^k, s_2^k successively eliminating b for more and more types of each bidder. We claim that the sequences s_1^k and s_2^k converge to s_1^* and s_2^* respectively, and thus that these are the the only types for whom b is not eliminated after infinitely many rounds of elimination. Indeed, if $v_1(s_1^k, s_2^{k-1}) = v_2(s_1^k, s_2^{k+1}) = b$ for every k, and if (s_1^k, s_2^k) were any convergent subsequence of these bounds with limit $\hat{s} = (\hat{s}_1, \hat{s}_2)$, then by continuity of the valuation functions $v_1(\hat{s}) = v_2(\hat{s}) = b$, i.e., $\hat{s} = s^*$.

Since this argument applies to every bid $v(s)$ for s on the pivot curve, and since we have already eliminated all other bids for every type of both bidders, after iterative elimination of ex-post weakly dominated strategies there remains a unique bidding strategy for each bidder i, namely the efficient ex-post equilibrium in which s_i bids $b_i(s_i)$.

The formal proof below treats the asymmetric case and also shows that not only this elimination sequence, but every vigilant elimination sequence leads to the efficient allocation.

3.2. Proof of Theorem 3.1

We consider the following generalized VCG mechanism. Each bidder i's message space is [0,1]. The set of reporting strategies is the set R_i of maps $\rho_i : [0,1] \to [0,1]$. Given the profile of reports r, bidder i's imputed valuation is $v_i(r)$, and the object is awarded to the bidder with the highest imputed valuation. Ties are broken by randomization. The winning bidder i makes a payment to the auctioneer equal to his *pivotal valuation*: $v_i(s_i^*, r_{-i})$ where $s_i^* = \min\{s_i \in [0,1] : v_i(s_i, r_{-i}) \geq v_{-i}(r_{-i}, s_i)\}$. The loser makes no transfer. As is well-known, the truthful reporting strategies form an ex-post equilibrium of this mechanism whose outcome is ex-post efficient.

We begin by introducing some notation. Consider the set S of type pairs s such that $v_1(s) = v_2(s)$. If $S = \emptyset$, then by the intermediate value theorem (hereafter IVT), either $v_1 > v_2$ for every type profile or $v_2 > v_1$ for every type profile. In either case, every strategy profile brings about the unique efficient allocation. In particular, no strategy is dominated, R is the unique dominance solution and we are done. So assume $S \neq \emptyset$, and let $S_i \subset [0,1]$ be the projection of this set into bidder i's set of types.

Lemma 3.2. *1. S_i is an interval $[\underline{s}_i, \bar{s}_i]$.*

2. If $s_i < \underline{s}_i$ then $v_i(s_i, \hat{s}_{-i}) < v_{-i}(\hat{s}_{-i}, s_i)$ for every $\hat{s}_{-i}$.

3. If $s_i > \bar{s}_i$ then $v_i(s_i, \hat{s}_{-i}) > v_{-i}(\hat{s}_{-i}, s_i)$ for every $\hat{s}_{-i}$.

4. If $\underline{s}_i > 0$ then $\underline{s}_{-i} = 0$.

5. If $\bar{s}_i < 1$ then $\bar{s}_{-i} = 1$.

Proof. Suppose $s_i \notin S_i$. There exists no $s_{-i} \in [0,1]$ such that $v_i(s_i, s_{-i}) = v_{-i}(s_{-i}, s_i)$. Then either $v_i(s_i, s_{-i}) - v_{-i}(s_{-i}, s_i) > 0$ for every s_{-i}, or the opposite inequality holds for every s_{-i}. This is a consequence of the intermediate value theorem (hereafter IVT) and the continuity of v_i and v_{-i}.

The single crossing condition implies in the former case that for all $s'_i > s_i$, and every s_{-i}, $v_i(s'_i, s_{-i}) > v_{-i}(s_{-i}, s'_i)$, and hence $s'_i \notin S_i$. In the latter case the conclusion is that $s'_i \notin S_i$ for every $s'_i < s_i$. This argument shows that the complement of S_i is of the form $[0, \underline{s}_i) \cup (\bar{s}_i, 1]$, and hence that S_i is an interval.

To prove the second part, note that $v_i(\underline{s}_i, \underline{s}_{-i}) = v_{-i}(\underline{s}_{-i}, \underline{s}_i)$. By the single-crossing condition $v_i(s_i, \underline{s}_{-i}) < v_{-i}(\underline{s}_{-i}, s_i)$, since $s_i < \underline{s}_{-i}$. There could not be any $\hat{s}_{-i}$ such that $v_i(s_i, \hat{s}_{-i}) > v_{-i}(\hat{s}_{-i}, s_i)$ otherwise the IVT would imply the existence of a s'_{-i} such that $v_i(s_i, s'_{-i}) = v_{-i}(s'_{-i}, s_i)$ which is impossible since $s_i < \underline{s}_i$. The third part is proven by a similar argument.

The fourth and fifth claims follow immediately from the second and third. □

In view of this result, we will write $S_i = [\underline{s}_i, \bar{s}_i]$ (by the continuity of the valuation functions, S_i is closed). For each $s_i \in S_i$, we denote by $b_i(s_i)$ the unique value b such that there exists a $s_{-i} \in T_{-i}$ for which $v_i(s_i, s_{-i}) = v_{-i}(s_{-i}, s_i) = b$. Observe that $b_{-i}(r_{-i})$ is the generalized VCG payment bidder i would have to make if he were to win the auction against a report of $r_{-i} \in S_{-i}$.[9] Furthermore $b_1(\underline{s}_1) = b_2(\underline{s}_2) := \underline{b}$ and $b_1(\bar{s}_1) = b_2(\bar{s}_2) := \bar{b}$. From the continuity and monotonicity of the valuation functions, $b_i(\cdot)$ is a continuous, increasing bijection between S_i and $B := [\underline{b}, \bar{b}]$. We now extend b_i to all of $T_i = [0,1]$, by specifying $b_i(s_i) = v_i(s_i, 1)$ for $s_i > \bar{s}_i$ and $b_i(s_i) = v_i(s_i, 0)$ for $s_i < \underline{s}_i$.

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9 In fact, the value $b_i(s_i)$ is the bid that type s_i would make in the efficient ex post equilibrium of the second-price auction. Thus, our arguments below that a report r_i is dominated for a type s_i, are equivalent to showing that, in the second-price auction, the bid $b_i(r_i)$ is dominated for type s_i.

We now introduce the following relation, which is central to the argument. Given $b \in B$, and $s_{-i} \in T_{-i}$, implicitly define $\varphi_i(s_{-i}, b) \in [0,1]$ by

$$v_i(\varphi_i(s_{-i}, b), s_{-i}) = b$$

where this exists. Note that when such a type exists, it is unique by the strict monotonicity of v_i.

Lemma 3.3. *1. $\varphi_i(s_{-i}, \cdot)$ is continuous on its domain of definition.*

2. $\varphi_i(\cdot, b)$ is decreasing on its domain of definition.

3. If $s_i = \varphi_i(s_{-i}, b)$ exists, then $\varphi_{-i}(s_i, b)$ exists.

4. For $i = 1,2$, for each $b \in B$, either $\varphi_i(1,b)$ or $\varphi_{-i}(0,b)$ exists.

Proof. The first two claims follow from our assumed properties of v_i: continuity in the first case, monotonicity in the second case.

To prove the third, assume first $v_{-i}(s_{-i}, s_i) \leq b$. Then $v_i(s_i, s_{-i}) \geq v_{-i}(s_{-i}, s_i)$ and therefore $s_i \geq \underline{s}_i$ by part 2 of Lemma 3.2. If $s_i \geq \bar{s}_i$, then by monotonicity $v_{-i}(\bar{s}_{-i}, s_i) \geq v_{-i}(\bar{s}_{-i}, \bar{s}_i) \geq b$. On the other hand, if $s_i < \bar{s}_i$, then there exists $\hat{s}_{-i}$ such that $v_i(s_i, \hat{s}_{-i}) = v_{-i}(\hat{s}_{-i}, s_i)$. By the single-crossing condition, since $v_{-i}(s_{-i}, s_i) - v_i(s_i, s_{-i}) \leq 0$ we must have $\hat{s}_{-i} \geq s_{-i}$. By monotonicity, $v_i(s_i, \hat{s}_{-i}) \geq v_i(s_i, s_{-i}) = b$, hence $v_{-i}(s_{-i}, s_i) \geq b$. In either case there exists $\hat{s}_{-i}$ such that $v_{-i}(\hat{s}_{-i}, s_i) \geq b$ and the IVT now implies the existence of $s'_{-i} \in [s_{-i}, \hat{s}_{-i}]$ for which $v_{-i}(s'_{-i}, s_i) = b$. Thus $\varphi_{-i}(s_i, b) = s'_{-i}$. The case of $v_{-i}(s_{-i}, s_i) \geq b$ is handled by first noting that this implies $s_i \leq \bar{s}_i$ and applying the analogous argument.

Finally, for the fourth claim, suppose for bidder 2, say, that $\varphi_2(1,b)$ does not exist, i.e. there is no s_2 such that $v_2(1, s_2) = b$. Since $b \in B$ we know that $v_2(1,1) \geq b$, and hence by the IVT we must have $v_2(1, s_2) > b$ for every s_2. In particular, $v_2(1,0) > b$.

Observe that $v_1(\underline{s}_1, \underline{s}_2) = v_2(\underline{s}_1, \underline{s}_2)$, and hence by the single-crossing condition, $v_1(\underline{s}_1, 0) \geq v_2(\underline{s}_1, 0)$ since $0 \leq \underline{s}_2$. Applying the single-crossing condition again, we see that $v_1(1,0) \geq v_2(1,0) > b$. And since $v_1(0,0) \leq b$, the IVT yields a s_1 such that $v_1(s_1, 0) = b$, and $\varphi_1(0,b) = s_1$. □

Define the following pair of functions.

$$m_i^0(b) = \begin{cases} \varphi_i(\bar{s}_{-i}, b) & \text{if it exists} \\ 0 & \text{otherwise} \end{cases}$$

$$M_i^0(b) = \begin{cases} \varphi_i(\underline{s}_{-i}, b) & \text{if it exists} \\ 1 & \text{otherwise} \end{cases}$$

It follows from parts 3 and 4 of Lemma 3.3 that for each $b \in B$ and $i = 1,2$, $\varphi_i(m_{-i}^0(b), b)$ and $\varphi_i(M_{-i}^0(b), b)$ exist. We can thus inductively define

$$\begin{aligned} M_i^k(b) &= \varphi_i(m_{-i}^{k-1}(b), b) \qquad k = 1,2,\dots \\ m_i^k(b) &= \varphi_i(M_{-i}^{k-1}(b), b) \qquad k = 1,2,\dots \end{aligned}$$

which by Lemma 3.3 yield continuous functions of b for each k.

Lemma 3.4. *For all $k \geq 1$ and for all $b \in B$,*

1. $s_i \geq$ *(resp.* $>$*)*$M_i^k(b) \implies v_i(s_i, s_{-i}) \geq$ *(resp.* $>$*)* $b\ \forall s_{-i} \geq m_{-i}^{k-1}(b)$.
2. $s_i \leq$ *(resp.* $<$*)*$m_i^k(b) \implies v_i(s_i, s_{-i}) \leq$ *(resp.* $<$*)* $b\ \forall s_{-i} \leq M_{-i}^{k-1}(b)$.

Proof. We prove the first claim. The second is shown by the symmetric argument. If $s_i > M_i^k(b)$ and $s_{-i} \geq m_{-i}^{k-1}(b)$, then by the monotonicity of v_i, $v_i(s_i, s_{-i}) \geq v_i(s_i, m_{-i}^{k-1}(b)) > v_i(M_i^k(b), m_{-i}^{k-1}(b))$ and the latter is equal to b by definition. □

We wish first to show that for each $b = b_i(s_i) \in B$, the sequences $M_i^k(b)$ and $m_i^k(b)$ for $i = 1,2$, converge. By definition, $M_{-i}^0(b) \leq \bar{s}_{-i}$ and $m_{-i}^0(b) \geq \underline{s}_{-i}$. Hence by the monotonicity of φ_i (part 2 of Lemma 3.3),

$$M_i^1(b) = \varphi_i(m_{-i}^0(b), b) \leq M_i^0(b)$$

because the latter is either $\bar{s}_i$ or $\varphi_i(\underline{s}_{-i}, b)$. Similarly $m_i^1(b) \geq m_i^0(b)$.

Now by the monotonicity of φ_i, we can inductively conclude that $m_i^k(b) \geq m_i^{k-1}(b)$ and $M_i^k(b) \leq M_i^{k-1}(b)$ for all $k \geq 1$. Furthermore, since the range of φ_i is S_i, these sequences are bounded. Hence for $i = 1,2$, there are types $m_i^*(b)$ and $M_i^*(b)$ such that $m_i^k(b) \to m_i^*(b)$ and $M_i^k(b) \to M_i^*(b)$.

Finally, since $v_i(m_i^k(b), M_{-i}^{k-1}(b)) = v_i(M_i^k(b), m_{-i}^{k-1}(b)) = b$, for all $k \geq 1$, the continuity of the valuation functions implies that for each i,

$$v_i(m_i^*(b), M_{-i}^*(b)) = v_i(M_i^*(b), m_{-i}^*(b)) = b$$

which in turn implies $m_i^*(b) = M_i^*(b) = s_i$.

To summarize the argument to this point, for each $s_i \in S_i$,

$$\begin{aligned} M_i^k(b_i(s_i)) &\downarrow s_i \\ m_i^k(b_i(s_i)) &\uparrow s_i \end{aligned} \tag{1}$$

We now describe an elimination sequence which yields an efficient dominance solution. As a first step, we eliminate any strategy ρ such that for some $s_i < \bar{s}_i$, $\rho(s_i) > \bar{s}_i$ or for some $s_i > \underline{s}_i$, $\rho(s_i) < \underline{s}_i$. This can be done in the former case because a report $r_i > \bar{s}_i$ will win the auction against any report of the opponent, in particular, against a report $r_{-i} = \bar{s}_{-i}$. In this case, the payment would be $b_i(\bar{s}_i) = v_i(\bar{s}_i, \bar{s}_{-i}) > v_i(s_i, \hat{s}_{-i})$ for any type $\hat{s}_{-i}$ of the opponent. The alternative report $\bar{s}_i$ for type s_i would affect the allocation only in this case, and in this case would win with a lower probability. Since the net payoff from winning in this case was negative, this is a strict improvement. To summarize, for every type of the opponent, and for every report of the opponent, the report $\bar{s}_i$ does no worse than r_i, and does strictly better whenever the opponent reports $\bar{s}_{-i}$. A similar argument shows that a report of $r_i < \underline{s}_i$ is dominated for a type $s_i > \underline{s}_i$ by the report $\underline{s}_i$.

Next, for each $k = 1, 2, \ldots$, define the following subsets of $S_i \times S_i$:

$$\begin{aligned} D_i^k &= \{(s_i, r_i) \in [0,1] \times S_i : s_i > M_i^k(b(r_i))\} \\ C_i^k &= \{(s_i, r_i) \in [0,1] \times S_i : s_i < m_i^k(b(r_i))\} \end{aligned}$$

We observe that any such pair of sets describes a set of admissible bidding strategies for bidder i, namely, the set of bidding strategies $\rho(\cdot)$ such that for each $s_i \in S_i$,

$$(s_i, \rho(s_i)) \notin C_i^k \cup D_i^k$$

Let β_i^k denote the set bidding strategies represented in this sense by the sets C_i^k and D_i^k, and set $\beta^{-1} = R$.

Lemma 3.5. *For each $k \geq 0$, every strategy not in β_i^k, is dominated within β^{k-1}, by a strategy in β_i^k.*

Proof. Consider any $(s_i, r_i) \in D_i^k$. Define

$$Z = \{b_i(\hat{r}_i) > b_i(r_i) : s_i \leq M_i^k(b_i(\hat{r}_i))\}.$$

By the continuity of b_i and M_i^k, Z is closed. Suppose $Z \neq \emptyset$. Then the value $b_i(r_i^*) = \min Z$ is well-defined, and by continuity $s_i = M_i^k(b_i(r_i^*)) \geq m_i^k(b_i(r_i^*))$ and thus $(s_i, r_i^*) \notin C_i^k \cup D_i^k$. We claim that reporting r_i^* ex post dominates reporting r_i for s_i.

To prove this, note that the report of r_i^* changes the allocation relative to that which would obtain under r_i if and only if bidder $-i$ reports some r_{-i} where

$$b_{-i}(r_{-i}) \in [b_i(r_i), b_i(r_i^*)]$$

Note first that in all such cases, by reporting r_i^* i wins the auction and pays $b_{-i}(r_{-i})$ rather than losing the auction and paying nothing, as he would if he were to report r_i. And second, observe that given the strategies β_{-i}^{k-1} for $-i$ that remain, a type s_{-i} will report r_{-i} only if $s_{-i} \geq m_{-i}^{k-1}(b_{-i}(r_{-i}))$.

Consider any such $b_{-i}(r_{-i})$. Suppose $s_i < M_i^k(b_{-i}(r_{-i}))$. Then since $b_{-i}(r_{-i}) \in [b_i(r_i), b_i(r_i^*)]$, there is some type $\hat{s}_i \in [r_i, r_i^*]$ such that $b_{-i}(r_{-i}) = b_i(\hat{s}_i)$ and thus

$$s_i < M_i^k(b_i(\hat{s}_i)). \tag{2}$$

Now if $\hat{s}_i = r_i$, then $b_i(r_i) = b_i(\hat{s}_i) = b_{-i}(r_{-i})$ and thus $M_i^k(b_i(r_i)) = M_i^k(b_{-i}(r_{-i})) > s_i$, which is a contradiction since $(s_i, r_i) \in D_i^k$. So $\hat{s}_i \in (r_i, r_i^*]$ and thus $b_i(\hat{s}_i) \in (b_i(r_i), b_i(r_i^*)]$. Now by our supposition (2) and the definition of r_i^* we must have $b_i(\hat{s}_i) \leq b_i(r_i^*)$, so in fact $b_i(\hat{s}_i) = b_i(r_i^*)$. But this implies $s_i = M_i^k(b_i(\hat{s}_i))$, which contradicts (2). We thus conclude that our original supposition was false and $s_i \geq M_i^k(b_{-i}(r_{-i}))$ for all $b_{-i}(r_{-i}) \in [b_i(r_i), b_i(r_i^*)]$.

By Lemma 3.4, it follows that for all $b_{-i}(r_{-i}) \in [b_i(r_i), b_i(r_i^*)]$, $v_i(s_i, s_{-i}) \geq b_{-i}(r_{-i})$ for every $s_{-i} \geq m_{-i}^{k-1}(b_{-i}(r_{-i}))$. Furthermore, since $s_i > M_i^k(b_i(r_i))$, Lemma 3.4 implies that the inequality is strict for $b_{-i}(r_{-i}) = b_i(r_i)$. We have just shown that for all reports of $-i$ such that reporting r_i^* wins the object where reporting r_i would not, and for all types of $-i$ that could make such reports in β_{-i}^{k-1}, the value of the object exceeds its price, strictly in at least one case.. Thus, the report r_i^* ex post dominates r_i for s_i.

Now suppose $Z = \emptyset$. Then applying the same argument as above, $s_i > M_i^k(b)$ for all $b \in [b_i(r_i), \bar{b}]$, implying that $v_i(s_i, \hat{s}_{-i}) > b$ for all such b for every possible $\hat{s}_{-i}$ for which there remains a report r_{-i} such that $b = b_{-i}(r_{-i})$. Since truthful reporting is never eliminated, one such possibility is $\bar{s}_{-i}$, reporting $\bar{b}$. In this case $v_i(s_i, \bar{s}_{-i}) > \bar{b} = v_i(\bar{s}_i, \bar{s}_{-i})$ and so monotonicity implies $s_i > \bar{s}_i$. In this case, r_i is dominated for s_i by the truthful report s_i. This report changes the allocation only in the above mentioned cases, where it ensures that i wins the auction an pays $b_{-i}(r_{-i})$, a price strictly less than his valuation.

In both cases, the dominating reports r_i^* satisfy $(s_i, r_i^*) \notin D_i^k \cup C_i^k$. By a similar argument, it can be shown that for any (s_i, r_i) in C_i^k, the report r_i is dominated for s_i by an alternative report r_i^* for which $(s_i, r_i^*) \notin D_i^k \cup C_i^k$.

We have thus shown that for any type $s_i \in S_i$, if $(s_i, r_i) \in C_i^k \cup D_i^k$, then r_i is ex post dominated for s_i by an alternative report that is not in $C_i^k \cup D_i^k$. It follows that all bidding strategies other than those in β_i^k are dominated by a strategy in β_i^k and thus can be eliminated. □

We can now conclude the proof of the existence of an efficient dominance solution of the generalized VCG mechanism. Let β be the set of strategy profiles remaining after the elimination sequence β^k. A reporting strategy ρ_i remains in β_i for bidder i iff

$$\rho_i(s_i) \in \begin{cases} (\bar{s}_i, 1] & \text{for } s_i > \bar{s}_i \\ [\bar{s}_i, 1] & \text{for } s_i = \bar{s}_i \\ \{s_i\} & \text{for } s_i \in S_i \\ [0, \underline{s}_i] & \text{for } s_i = \underline{s}_i \\ [0, \underline{s}_i) & \text{for } s_i < \underline{s}_i \end{cases}$$

Thus, for any profile of strategies within β, the allocation is identical, except possibly in the event that $s = \bar{s}$ or $s = \underline{s}$. However, in either of these two cases, the two bidders have the same valuation for the object, and for any strategy profile in β, the winner will pay that valuation. Thus, no strategies in β are dominated within β. Note also that β is an ex post equilibrium. Moreover, the allocation is ex post efficient. Finally, this elimination sequence is vigilant because every dominated strategy is eliminated at every stage.

We now establish that every dominance solution is an efficient ex post equilibrium. We will show that every vigilant dominance solution is a subset of β. Let $\hat{\beta}^k$ be an vigilant elimination sequence. Then there exists K such that any strategy that is dominated in K consecutive stages must be eliminated. We define

$$\hat{\beta}_i^k(s_i) = \{\rho_i(s_i) : \rho_i \in \hat{\beta}_i^k\}.$$

We will make use of the following lemma:

Lemma 3.6. *For every $s_i \in S_i$*

1. *If $b = b_{-i}(s_{-i}) > b_i(s_i)$, then $v_i(s_i, s_{-i}) < b$.*

2. *If $b = b_{-i}(s_{-i}) < b_i(s_i)$, then $v_i(s_i, s_{-i}) > b$.*

Proof. Consider the first claim. There is a $\hat{s}_i > s_i$ such that $v_i(\hat{s}_i, s_{-i}) = b_{-i}(s_{-i})$. By monotonicity, $v_i(s_i, s_{-i}) < b_{-i}(s_{-i}) = b$. The second claim follows from the symmetric argument. □

The first step is to show by induction that no elimination sequence can eliminate a strategy ρ_i specifying $\rho_i(s_i) = s_i$ for all $s_i \in S_i$, $\rho_i(s_i) > \bar{s}_i$ for all $s_i > \bar{s}_i$ and $\rho_i(s_i) < \underline{s}_i$ for all $s_i < \underline{s}_i$. We will call any such strategy a "truthtelling" strategy.

Obviously all truthtelling strategies belong to $\hat{\beta}^0$. Now consider any stage of elimination k such that for each i all truthtelling strategies remain in β^{k-1}. First consider any untruthful report r_i for type $s_i \in (\underline{s}_i, \bar{s}_i)$ of bidder i. If $r_i > s_i$, then $b_i(r_i) > b_i(s_i)$ and whenever $-i$ uses a truthtelling strategy ρ_{-i}, and his type s_{-i} satisfies $b_{-i}(s_{-i}) \in (b_i(s_i), b_i(r_i))$, bidder i will win the auction and pay $b_{-i}(s_{-i})$. By Lemma 3.6, $b_{-i}(s_{-i}) > v_i(s_i, s_{-i})$. Since i would have lost the auction in these cases with report s_i, truthtelling does strictly better. On the other hand, if $r_i < s_i$, then for any $b_{-i}(r_{-i}) \in (b_i(r_i), b_i(s_i))$, bidder i would lose the auction with r_i whereas he would have won with s_i and payed $b_{-i}(s_{-i})$. In these cases, given truthelling by $-i$, i's payoff from winning would be strictly positive by Lemma 3.6, so again s_i strictly prefers to tell the truth. We have shown that there is at least one possible case in $\hat{\beta}^{k-1}$ namely truthtelling by the opponent, such that $s_i \in (\underline{s}_i, \bar{s}_i)$ strictly prefers to tell the truth. Thus, truthtelling by such types cannot be dominated. A slight modification of this argument delivers the same conclusion for types $\underline{s}_i$ and $\bar{s}_i$. (For example, a report greater than $\bar{s}_i$ cannot dominate $\bar{s}_i$ because either report always leads to the same outcome.)

Next, given that truthelling is an element of $\hat{\beta}^{k-1}$, reports $r_i > \bar{s}_i$ for types $s_i > \bar{s}_i$ and $r_i < \underline{s}_i$ for types $s_i < \underline{s}_i$ cannot be eliminated. To see this note that if, say , type $s_i > \bar{s}_i$ reports $\hat{r}_i \leq \bar{s}_i$ rather than $r_i > \bar{s}_i$ (other alternatives can never change the allocation), then in the event that the opponent's type is $\bar{s}_{-i}$ (which is 1 in this case) and reports truthfully, i will lose the auction with $\hat{r}_i$ rather than win with r_i and pay $\bar{b} = v_i(\bar{s}_i, \bar{s}_{-i}) < v_i(s_i, \bar{s}_{-i})$. Since there is at least one case in which r_i leads to a strictly lower payoff than r_i, we conclude that $\hat{r}_i$ cannot dominate r_i. A similar argument shows that no reports below $\underline{s}_i$ can be dominated for a type $s_i < \underline{s}_i$.

We have shown that no truthtelling strategy can be eliminated in any round. Thus, every dominance solution $\hat{\beta}$ is non-empty and includes all truthtelling strategies. We will now show that $\hat{\beta} \subset \beta$. To do so, we will show that there is no strategy ρ_i in $\hat{\beta}$ satisfying any of the following:

1. $\rho_i(s_i) = \hat{s}_i \in S_i$ for some $s_i \neq \hat{s}_i$
2. $\rho_i(s_i) > \bar{s}_i$ for some $s_i < \bar{s}_i$
3. $\rho_i(s_i) < \underline{s}_i$ for some $s_i > \underline{s}_i$

To show that 2 cannot hold for any ρ_i in $\hat{\beta}$ observe that the domination argument used in the first round of elimination leading to β in the proof of Lemma 3.5 applies in any round in which all truthtelling strategies survive. Thus, if $s_i < \bar{s}_i$ and $r_i > \bar{s}_i$, then r_i is dominated for s_i by $\bar{s}_i$ in every round.

Suppose r_i is not eliminated in any round for type s_i. Then by vigilance, it must be the case that in some round $k \leq K$, reporting $\bar{s}_i$ is eliminated for type s_i. Let k be the first round in which reporting $\bar{s}_i$ is eliminated for type s_i. Then there is some report $\hat{r}_i$, which survives for type s_i in round k and dominates reporting $\bar{s}_i$ within $\hat{\beta}^{k-1}$. This means that for every $b \in [b_i(\hat{r}_i), b_i(\bar{s}_i)]$, and for every s_{-i} such that $b \in b_{-i}(\hat{\beta}^{k-1}_{-i}(s_{-i}))$, we have $v_i(s_i, s_{-i}) \leq b$. This implies, by lemma 3.6 and the fact that all truthtelling strategies remain, that $\hat{r}_i \geq s_i$. Applying lemma 3.6 again, we conclude that for every b in the interior of this interval, $v_i(s_i, s_{-i}) < b$ for $b_{-i}(s_{-i}) = b$. Since the report of $\hat{r}_i$ changes the allocation relative to r_i only when $b_{-i}(r_{-i}) \in [b_i(\hat{r}_i), b_i(\bar{s}_i)]$ and in all such cases the payoff to bidder i is non-positive and in some cases strictly negative, $\hat{r}_i$ dominates r_i as well. We can iterate this argument to conclude that for every k, r_i is dominated within $\hat{\beta}^{k-1}$ by an element of $\hat{\beta}^k$. Therefore, by vigilance, $r_i \notin \hat{\beta}_i(s_i)$. An analogous argument shows that 3 cannot hold.

Finally, we show that 1 cannot be true of any ρ_i in $\hat{\beta}$. We will do this by demonstrating that for every k, every $(s_i, r_i) \in D_i^k \cup C_i^k$ is eventually eliminated. The argument is by induction. Let $(s_i, r_i) \in D_i^1 \cup C_i^1$. Then r_i is dominated within $\hat{\beta}^0$ for s_i by some $r_i^0 \neq r_i$. By the same argument as above, r_i^0 dominates r_i for s_i within $\hat{\beta}^k$ for every k.

Suppose $r_i^0 \in \hat{\beta}_i^K(s_i)$. Then by vigilance, since r_i could be eliminated in K consecutive stages, r_i must be eliminated for s_i in $\hat{\beta}_i^K$. On the other hand, if $r_i^0 \notin \hat{\beta}_i^K(s_i)$, then there is a sequence of reports $r_i^0, r_i^1, \ldots, r_i^T$, such that $r_i^T \in \hat{\beta}_i^K(s_i)$, and for each $t = 0, \ldots, T-1$, there is a stage $k(t)$ such that r_i^t was eliminated in stage $k(t)$ because it was dominated within $\hat{\beta}^{k(t)}$ by $r_i^{t+1} \in \hat{\beta}^{k(t)+1}$. We can now argue just as above that r_i^0 is dominated successively by each r_i^t, and hence that r_i^0 could have been eliminated for K consecutive stages. By vigilance, $r_i^0 \notin \hat{\beta}_i^K(s_i)$. Thus all reporting strategies in β_i^0 will be eliminated by stage K.

Now suppose that for a given $\bar{k}$, all strategies in $\beta^0 \setminus \beta^{\bar{k}}$ are eliminated by round $\bar{k}K$, and let $(s_i, r_i) \in C_i^{\bar{k}+1} \cup D_i^{\bar{k}+1}$. By lemma 3.5, r_i is dominated within $\hat{\beta}^{\bar{k}K}$ for s_i by some r_i^0. If $r_i^0 \in \hat{\beta}^{(\bar{k}+1)K}$, then r_i could be eliminated in each of the K intervening stages and by vigilance would be eliminated in stage $(\bar{k}+1)K$.

If $r_i^0 \notin \hat{\beta}^{(\bar{k}+1)K}$, then there exists a sequence of reports $r_i^0, r_i^1, \ldots, r_i^T$, corresponding to stages $k(t)$ for $t = 0, \ldots, T-1$ with $r_i^T \in \hat{\beta}^{(\bar{k}+1)K}$ and with each r_i^t eliminated in stage $k(t)$, dominated within $\hat{\beta}^{k(t)}$ by $r_i^{t+1} \in \hat{\beta}^{k(t)+1}$. Then by the same transitivity argument used previously, r_i^0 is dominated successively by each r_i^t and hence r_i could be eliminated in each stage from $\bar{k}K+1$ to $(\bar{k}+1)K$ and hence by vigilance must be eliminated in the latter stage.

Since (s_i, r_i) was arbitrary, we have shown that all strategies within $\beta^{\bar{k}} \setminus \beta^{\bar{k}+1}$ must be eliminated by stage $(\bar{k}+1)K$, and this concludes the inductive step and the proof.

4. MULTI-UNIT AUCTIONS WITH TWO BIDDERS

The essential equivalence between implementability in ex post equilibrium and ex post dominance implementability of efficient allocation generalizes straightforwardly to multi-unit auctions with two bidders. Suppose there are $L \geq 1$ identical objects. For any $i \in \mathscr{I}$ and $1 \leq l \leq L$, let $v_i^l : T \to \mathbf{R}_+$ be bidder i's marginal valuation of the lth object he wins. We assume non-increasing marginal valuations: $[1 \leq l < l' \leq L] \Longrightarrow [v_i^l \geq v_i^{l'}]$. We maintain the assumption that v_i^l is continuous, strictly increasing, and satisfies the following single-crossing condition:

Assumption 2 (Single-Crossing Property)**.** *For any $1 \leq l \leq L$, for any $i \neq j$ and $\hat{s}_j$, the difference*

$$g(s_i) = v_i^l(s_i, \hat{s}_j) - v_j^{L+1-l}(s_i, \hat{s}_j)$$

crosses zero at most once, and from below.

Theorem 4.1. *When there are two bidders and $L \geq 1$ identical objects, the efficient allocation is ex post dominance implementable.*

Proof: Label the L objects from 1 to L. Consider the following indirect version of the generalized VCG mechanism. We imagine L simultaneous auctions. For each auction l there will correspond sets S_i^l and $B^l = [\underline{b}^l, \bar{b}^l]$ and mappings $b_i^l(\cdot)$ constructed just as in the single-unit case, now using the valuation functions $v_1^l(\cdot)$ and $v_2^{L-l+1}(\cdot)$. For notational convenience below, we will modify the definition of b_i^l outside of S_i^l by defining

$$\bar{b} = 1 + \max_{i,l,s} v_i^l(s)$$

$$\underline{b} = -1 + \min_{i,l,s} v_i(s)$$

and setting $b_i^l(s) = \bar{b}$ for $s_i > \bar{s}_i{}^l$ and $b_i^l(s) = \underline{b}$ for $s_i < \underline{s}_i{}^l$.

Each bidder simultaneously submits L reports $r_i^1, \ldots, r_i^L$. with the restriction that $b_1^1(r_1^1) \leq, \ldots \leq b_1^L(r_1^L)$ and $b_2^1(r_2^1) \geq \ldots \geq b_2^L(r_2^L)$. Let Δ_i denote the set of reporting strategies for i meeting the corresponding restriction. The idea is that for some $\hat{l}$, bidder 1 should win objects 1 through $\hat{l}$, and bidder 2 objects $\hat{l}+1$ through L. Thus object l is awarded to the bidder with the greater imputed valuation according to the reports r^l using the marginal valuation functions v_1^l, v_2^{L-l+1}. Ties will always be broken in favor of bidder 1[10] The payment of the winner of object l is calculated in the same way as the single unit VCG auction, using these reports and marginal valuation functions.

For each l and for each $k = 0, 1 \ldots$, we define the functions $m_{i,l}^k(\cdot)$ and $M_{i,l}^k(\cdot)$ on the sets B^l as before. By the same argument as in the single object case, these functions converge pointwise to e.g. $m_{1,l}^*(b)$ where $b_1^l(m_{1,l}^*(b)) = b$. Moreover,

Lemma 4.2. *For every $l < L$, and for $k \geq 0$*

1. $\underline{s}_1^l \leq \underline{s}_1^{l+1}$, $\bar{s}_1^l \leq \bar{s}_1^{l+1}$, $\underline{s}_2^l \geq \underline{s}_2^{l+1}$, *and* $\bar{s}_2^l \geq \bar{s}_2^{l+1}$.
2. $m_{1,l}^k(b) \leq m_{1,l+1}^k(b)$, $M_{1,l}^k(b) \leq M_{1,l+1}^k(b)$, $m_{2,l}^k(b) \geq m_{2,l+1}^k(b)$, $M_{2,l}^k(b) \geq M_{2,l+1}^k(b)$.

[10] This simplifies the notation in the proof. In general, any tie-breaking rule of the following form will suffice. Prior to receiving reports, the auctioneer chooses an integer $t \in \{0, \ldots, L\}$. It can be kept secret or revealed to the bidders. If there is a tie in auction l, then object l is awarded to bidder 1 if and only if $l \leq t$.

Proof. The first follows from the assumption of non-increasing marginal valuations. Non-increasing marginal valuations also implies that the second claim holds for $k = 0$. Finally, the non-increasing marginal valuations and the monotonicity of φ_i implies that the same inequalities are satisfied for every $k > 0$. $\square$

The first step of elimination is to delete all reporting strategies ρ_i such that for some l and for some $s_i \in (\underline{s_i}^l, \bar{s_i}^l)$, $\rho_i^l(s_i) \notin S_i^l$. For example, take $s_1 < \bar{s}_1^l$ and suppose $\rho_1^l(s_1) > \bar{s}_1^l$ so that $b_1^l(\rho_1^l(s_1)) = \bar{b}$. And suppose l is the smallest index for which these two inequalities are satisfied for s_1. We will show that $\rho_1(s_1)$ is dominated for s_1 by the following list of reports $\hat{r}_1$.

$$\hat{r_1}^\lambda = \begin{cases} \max\{r \leq \rho_1^\lambda(s_1) : b_1^\lambda(r) \leq \bar{b}^l\} & \lambda \geq l \\ \rho_1^\lambda(s_1) & \lambda < l \end{cases}$$

By construction, $\hat{r}_1 \in \Delta_1$. To show that it dominates $\rho_1(s_1)$, note that $\hat{r}_1$ yields a different allocation than $\rho_1(s_1)$ only in the event that bidder 2 submits a list of reports r_2 such that for some auction λ,

$$b_1^\lambda(\hat{r_1}^\lambda) \leq b_2^\lambda(r_2^\lambda) \leq b_1^\lambda(\rho_1^\lambda(s_1)) \tag{3}$$

with at least one of the inequalities strict. Notice that this can only be true for $\lambda \geq l$ since it is only in these auctions that bidder 1's report has changed. And in this case, since $\hat{r_1}^\lambda \neq \rho_1^\lambda(s_1)$ by definition we must have $b_1^\lambda(\hat{r_1}^\lambda) = \bar{b}^l$. Furthermore, for $\lambda < l$, $b_1^\lambda(\rho_1^\lambda(s_1)) \geq b_1^l(\rho_1^l(s_1)) = \bar{b}$, implying that bidder 1 will win the first $l-1$ objects for sure, and hence his marginal valuation for any additional object is no greater than v_1^l.

Now consider a λ satisfying (3). For any s_2, we have

$$v_1^\lambda(s) \leq v_1^l(s) < \bar{b}^l,$$

where the first inequality follows from the assumption of declining marginal valuations. To demonstrate the second inequality, observe that $\rho_1^l(s_1) > \bar{s}_1^l$ implies that $\bar{s}_1^l < 1$ so that $\bar{s}_2^l = 1$. Hence, $\bar{b}^l = v_1^l(\bar{s}_1, 1) > v_1^l(s)$ by monotonicity of v_1^l. Since $\bar{b}^l \leq b_2^\lambda(r_2^\lambda)$, and the latter is the price bidder 1 would have to pay for object λ , we have just shown that bidder 1's marginal valuation for any additional objects is strictly less than their price for any possible type of bidder 2. Thus, by lowering his reports to $\hat{r}_1$, against any strategy of 1 for which the allocation is altered, 1 strictly reduces the probability of earning a negative payoff. Thus, $\hat{r}_1$ ex post dominates $\rho_1(s_1)$ for s_1.[11]

By a similar argument we can eliminate any reporting strategy ρ_1 such that $\rho_1^l(s_1) < \underline{s}_1^l$ for some l and $s_1 > \underline{s}_1^l$. Such a reporting strategy will be dominated by one in which the report to the lth auction is raised to $\underline{s}_1^l$, with reports to all auctions $\lambda < l$ correspondingly raised in order to satisfy feasibility. This change affects the allocation only when $b_2^\lambda(r_2^\lambda) \leq \underline{b}^l$ for some $\lambda \leq l$, in which case 1 wins the auction and pays no more than $\underline{b}^l$. In this case, we can argue analogously to the above that bidder 1's valuation always strictly exceeds this price and hence the change only increases the probability of a positive payoff. The symmetric pair of arguments applies to bidder 2.

We construct the remainder of the elimination sequence as follows. For each $k = 0, 1, \ldots$, define the following nested sequence of strategy sets for each bidder i

$$\beta_i^k = \{\rho_i \in \Delta_i : b_i^l(\rho_i^l(s_i)) = b \in B^l \implies \\ s_i \in [m_{i,l}^k(b), M_{i,l}^k(b)] \qquad l = 1, \ldots, L\} \quad (4)$$

Let β^k be the set of profiles $\beta_1^k \times \beta_2^k$, and set $\beta^{-1} = \Delta$.

Lemma 4.3. *For each $k \geq 0$, every strategy not in β_i^k, is dominated within β^{k-1}, by a strategy in β_i^k.*

Proof. Consider first a reporting strategy $\rho_1$such that $\rho_1(s_1) = r_1$ for a type s_1 of bidder 1. Suppose for some auction λ,

$$b_1^\lambda(r_1^\lambda) \in B^\lambda \text{ but } s_1 < m_{1,\lambda}^k(b_1^\lambda(r_1^\lambda)). \quad (5)$$

And suppose l is the highest index λ for which this is true for s_1, r_1. Let

$$Z = \{\hat{r}_1{}^l \in [\underline{s}_1^l, r_1^l] : s_1 \geq m_{1,l}^k(b_1^l(r_1^l))\}$$

by the continuity of b_1^l and $m_{1,l}$, Z is closed.

Suppose $Z \neq \emptyset$. Then define $r_1^* = \max Z$, and consider the alternative list of reports $\hat{r}_1 = (r_1^1, \ldots, r_1^*, \ldots, r_1^L)$. We claim first that $\hat{r}_1 \in \Delta_1$.

[11] The report list $\hat{r}_1$ may itself be dominated. However, it is straightforward to extend the arguments from the single-unit case to find an undominated strategy that dominates ρ_1 for type s_1.

Suppose

$$b_1^{l+1}(r_1^{l+1}) > b_1^l(r_1^*) \tag{6}$$

Then

$$b_1^{l+1}(r_1^{l+1}) > \underline{b}. \tag{7}$$

Since r_1 was feasible,

$$b_1^l(r_1^l) \geq b_1^{l+1}(r_1^{l+1}). \tag{8}$$

It follows that $b_1^{l+1}(r_1^{l+1}) \leq \bar{b}^{l+1}$ otherwise $b_1^{l+1}(r_1^{l+1}) = \bar{b} > \bar{b}^l \geq b_1^l(r_1^l)$ which contradicts (8). Combining this with (7), we have shown that $b_1^{l+1}(r_1^{l+1}) \in B^{l+1}$. Thus, $b_1^{l+1}(r_1^{l+1})$ is well defined and

$$s_1 \geq m_{1,l+1}^k(b_1^{l+1}(r_1^{l+1})) \geq m_{1,l}^k(b_1^{l+1}(r_1^{l+1})) \tag{9}$$

where we have the first inequality by the definition of l and the second inequality is an application of lemma 4.2. But now (6), (8) and (9) contradict the definition of r_1^*. Thus $\hat{r}_1$ is feasible.

We now show that $\hat{r}_1$ dominates r_1 for s_1. First note that $\hat{r}_1$ changes the allocation relative to r_1 only if bidder 2's report r_2 satisfies

$$b_2^l(r_2^l) \in (b_1^l(\hat{r}_1{}^l), b_1^l(r_1^l)]$$

By construction of $\hat{r}_1$, we have $s_1 < m_{1,l}^k(b_2(r_2))$ for every such r_2. And in these cases, bidder 1 will lose the auction for object l rather than winning and paying price $b_2^l(r_2^l)$.

Since for any $\lambda \leq l$

$$b_1^\lambda(r_1^\lambda) \geq b_1^l(r_1^l) \geq b_2^l(r_2^l) \geq b_2^\lambda(r_2^\lambda)$$

bidder 1 will win[12] all objects $1, \ldots, l-1$. Furthermore, for every $\lambda > l$,

$$b_1^\lambda(r_1^\lambda) \leq b_1^l(\hat{r}_1{}^l) < b_2^l(r_2^l) \leq b_2^\lambda(r_2^\lambda)$$

bidder 1 will lose all objects $l+1, \ldots L$. Thus, the change from r_1 to $\hat{r}_1$ affects only the allocation of object l and bidder 1's marginal valuation for object l is given by $v_1^l(\cdot)$.

[12] Recall that ties are broken in favor of bidder 1.

For any reporting strategy ρ_2 for bidder 2 in the set β_2^{k-1}, if $\rho_2^l(s_2) = r_2^l$, then bidder 2's type s_2 must be no greater than $M_{2,l}^{k-1}(b_2^l(r_2^l))$. And since $s_1 < m_{1,l}^k(b_2^l(r_2^l))$, Lemma 3.4 implies that for every such s_2,

$$v_1^l(s_1,s_2) < b_2^l(r_2^l).$$

Thus, for any s_2 that would report r_2^l, the marginal payoff to bidder 1 from winning object l is strictly negative. Therefore in all cases in which $\hat{r}_1$ changes the allocation relative to r_1, it strictly increases the payoff of bidder 1.

Now suppose $Z = \emptyset$. Then $s_1 < m_{1,l}^k(b)$ for all $b \in [\underline{b}^l, b_1^l(r_1^l)]$. This implies in particular that $s_1 < \underline{s}_1^l$ because if $s_1 \geq \underline{s}_1^l$, then $s_1 \geq m_{1,l}^k(\underline{b}^l)$. And by Lemma 4.2, $s_1 < \underline{s}_1^\lambda$ for all $\lambda \geq l$. We can show by the same argument as above that for any report r_2 such that $b_2^l(r_2^l) \in [\underline{b}^l, b_1^l(r_1^l)]$, bidder 1 will win all objects $1,\ldots,l-1$, and for any type s_2 of bidder 2 such that the report r_2 is possible in β_2^{k-1}, bidder 1's marginal valuation for the lth object is strictly less than $b_2^l(r_2^l)$, the price.

Since for all $\lambda \geq l$, $m_{1,\lambda}^k(b) \geq m_{1,l}^k(b)$ and $b_2^\lambda(r_2^\lambda) \geq b_2^l(r_2^l)$, the payoff for all objects $\lambda > l$ is strictly negative as well. It follows that r_1 is dominated for s_1 by the report $\hat{r}_1$ defined as follows.

$$\hat{r}_1{}^\lambda = \begin{cases} r_1^\lambda & \text{if } \lambda < l \\ s_1 & \text{if } \lambda \geq l \end{cases}$$

Obviously $\hat{r}_1 \in \Delta_1$. And since $s_1 < \underline{s}_1^\lambda$, and hence $b_1^\lambda(\hat{r}_1{}^\lambda) = \underline{b}$ for all $\lambda \geq l$, by reporting $\hat{r}_1$, bidder 1 is guaranteed to lose auctions $l,\ldots,L$, and avoid the strictly negative payoff. Thus $\hat{r}_1$ dominates $\rho_1(s_1)$ for type s_1.

To complete the domination argument, let $\{\lambda_1,\ldots,\lambda_K\}$ be the set of indices of auctions for which (5) holds. By definition, $\lambda_K = l$, and we have just shown that $\rho_1(s_1)$ is dominated for s_1 by a list of reports, call it $\tilde{r}(K)$ for which (5) is not satisfied for any $\lambda \geq \lambda_K$. By repitition of the domination argument above, $\tilde{r}(K)$ is dominated by a report list $\tilde{r}(K-1)$ for which (5) is not satisfied for any $\lambda \geq \lambda_{K-1}$. By induction, we arrive at a list of reports $\tilde{r}(1)$ for which (5) is not satisfied for any auction, and which dominates $\tilde{r}(2)$ and by the transitivity of the dominance relation, dominates $\rho_1(s_1)$.

Now consider the set of auction indices μ for which

$$b_1^\mu(\tilde{r}^\mu(1)) \in B^\mu \text{ but } s_1 > M_{1,\mu}^k(b_1^\mu(\tilde{r}^\mu(1))). \tag{10}$$

Let l be the lowest index μ for which (10) is satisfied for $s_1, \tilde{r}(1)$. By an induction argument similar to the one in the previous paragraph, we can then show that $\tilde{r}(1)$, and by transitivity $\rho_1(s_1)$, is dominated by a report list, call it $\rho_1'(s_1)$ for which neither (5) nor (10) are satisfied for any auction.

Since s_1 was arbitrary, we have shown that for any $\rho_1$outside β_1^k, there is a ρ_1' in β_1^k which dominates it within β^{k-1}. □

We can now conclude the proof of theorem 4.1. Let β be the set of strategy profiles that remain after the elimination sequence β^k. A reporting strategy ρ_i remains in β_i for bidder i iff $\rho_i \in \Delta_i$ and, for all l,

$$\rho_i^l(s_i) \in \begin{cases} (\bar{s}_i{}^l, 1] & \text{for } s_i > \bar{s}_i{}^l \\ [\bar{s}_i{}^l, 1] & \text{for } s_i = \bar{s}_i{}^l \\ \{s_i\} & \text{for } s_i \in S_i^l \\ [0, \underline{s}_i{}^l] & \text{for } s_i = \underline{s}_i{}^l \\ [0, \underline{s}_i{}^l) & \text{for } s_i < \underline{s}_i{}^l \end{cases}$$

Clearly this is an efficient ex-post equilibrium. The proof that any vigilant elimination sequence leads to a subset of β follows lines identical to the single-unit case and is omitted. □

5. AUCTIONS WITH $N \geq 3$ BIDDERS

We have shown that when there are two bidders, the standard single crossing condition, essentially necessary for existence of an efficient ex post equilibrium, is sufficient for ex post dominance implementation. In this section, we show that with more than two bidders, conditions for ex post dominance implementability are generally strictly stronger than single crossing. To obtain easily interpretable necessary and sufficient conditions for ex post dominance implementation, we specialize in this section to symmetric linear valuation functions. Specifically, we assume the valuation functions take the following form.

$$\forall i \in \mathscr{I}, v_i = as_i + \sum_{j \neq i} s_j.$$

For this symmetric linear setting, there exists an efficient ex post equilibrium of the generalized VCG mechanism if and only if $a \geq 1$.Moreover, any direct revelation mechanism which has an efficient ex post equilibrium is a generalized VCG mechanism. Thus, to search for ex post dominance implementing direct mechanisms, it suffices to consider generalized VCG mechanisms.

The generalized VCG mechanism takes the following form. Each bidder i reports his type (i.e. $A_i = T_i$). If r is the profile of reports, each bidder's valuation is calculated assuming r was truthful, i.e. $v_i = v_i(r)$. If $v_i > v_j$ for each $j \neq i$, then i is awarded the object ($p_i(r) = 1$). In the event of ties, $p_i(r)$ can be chosen arbitrarily. In the symmetric linear setting with $a \geq 1$, the winner is the bidder reporting the highest type. The payment is determined as follows. Let bidder i be the winner of the object, and let $\hat{r}_i = \min\{s_i \in S_i \mid \forall j \in \mathscr{I}, v_i(s_i, r_{-i}) \geq v_j(s_i, r_{-i})\}$. Then $t_i(r) = v_i(\hat{r}_i, r_{-i}) + f_i(r_{-i})$ and $\forall j \neq i$, $t_j(r) = f_j(r_{-j})$, where f_i and the f_j's are arbitrary functions measurable only to r_{-i} and the r_{-j}'s, respectively.

The following proposition states that, in auctions with $n \geq 3$ bidders, the generalized VCG mechanism ex post dominance implements the efficient allocation only if a bidder's valuation is sufficiently sensitive to his own signal.

Proposition 5.1. *In the symmetric linear setting with $n \geq 3$ bidders, the generalized VCG mechanism ex post dominance implements the efficient allocation if and only if $a > n-1$.*

Proof. We first prove that, whenever $a > n-1$, truth-telling is a dominance solution of the generalized VCG mechanism with all the f_i's identically zero. Since truth-telling induces efficient allocation whenever $a > 1$, this will finish the proof of the "if" part.[13] We then prove that the generalized VCG mechanism, for any arbitrary f_i's, is not dominance solvable if $a \leq n-1$.

To begin with, we show that iteratively undominated reports must lie above some lower bound. Say that bidder i is pivotal at report r_j if i has reported the highest type and the second highest report is r_j.

Suppose that type s_i of bidder i is pivotal at report r_j. Then the maximum report among the other bidders is r_j, and i's payment is at most $(a+n-1)r_j$. Bidder i's payoff conditional on being pivotal at r_j is therefore at least $as_i - (a+n-1)r_j$. This payoff is zero for $r_j = \underline{r}(s_i)$ where

$$\underline{r}(s_i) = \frac{as_i}{a+n-1}$$

[13] We must also show that every dominance solution is an efficient ex post equilibrium. That part of the proof would be similar to the corresponding part of the proof of theorem 3.1.

Note also that this minimum payoff is strictly positive for all reports less than $\underline{r}(s_i)$. It follows that any report below the lower bound $\underline{r}(s_i)$ is dominated by $\underline{r}(s_i)$. Note that $\underline{r}(0) = 0$.

Now, assuming that each bidder's reporting strategy is bounded below by some linear function $\underline{r}(\cdot)$ whose slope is between $\frac{a}{a+n-1}$ and 1, and with inverse $\underline{s}(\cdot)$ we will construct an upper bound $\bar{r}(\cdot)$.

First, observe that for any report r_j, no type greater than $\underline{s}(r_j)$ will report r_j. Suppose that under report profile r, type s_i of bidder i is pivotal at report r_j. Bidder i's payoff is

$$as_i + \sum s_k - ar_j - \sum_{k \neq i} r_k$$

Since $s_j \leq \underline{s}(r_j)$, this payoff is no greater than

$$as_i + \underline{s}(r_j) - (a+1)r_j + (n-2) \max_{\hat{r}_k \leq r_j} (\underline{s}(\hat{r}_k) - \hat{r}_k)$$

Since the slope of $\underline{s}(\cdot)$ is greater than 1, the maximum is

$$as_i + (n-1)\underline{s}(r_j) - (a+n-1)r_j$$

By assumption, $a > n-1$. Therefore, the slope in r_j of the right hand expression is at most $\frac{(n-1)(a+n-1)}{a} - (a+n-1)$ which is negative. Letting $\bar{r}(s_i)$ be the r_j for which this expression is zero, we can implicitly solve for $\bar{r}(s_i)$

$$\bar{r}(s_i) = \frac{as_i + (n-1)\underline{s}(\bar{r}(s_i))}{a+n-1} \tag{11}$$

and conclude that any report greater than $\bar{r}(s_i)$ is dominated for type s_i by the report $\bar{r}(s_i)$.

Since $\underline{s}(\cdot)$ is linear, so will be $\bar{r}(s_i)$. And because the slope of $\underline{s}(\cdot)$ is greater than 1, it follows that the slope of $\bar{r}(s_i)$ is also greater than 1.

Now, we assume that all iteratively undominated reporting strategies are bounded above by some linear $\bar{r}(\cdot)$ with slope greater than 1, with inverse $\bar{s}(\cdot)$, and derive a new lower bound.

Suppose that under report profile r, type s_i of bidder i is pivotal at report r_j. Bidder i's payoff is at least

$$as_i + \bar{s}(r_j) - (a+1)r_j + (n-2) \min_{\hat{r}_k \leq r_j} (\bar{s}(\hat{r}_k) - \hat{r}_k)$$

Since the slope of $\bar{s}(\cdot)$ is less than 1, the minimum is

$$as_i + (n-1)\bar{s}(r_j) - (a+n-1)r_j$$

The slope in r_j is negative, so we can set this expression equal to zero, solve for $\underline{r}(s_i)$:

$$\underline{r}(s_i) = \frac{as_i + (n-1)\bar{s}(\underline{r}(s_i))}{a+n-1} \tag{12}$$

and conclude that any report less than $\underline{r}(s_i)$ is dominated for type s_i by the report $\underline{r}(s_i)$. If the slope of $\bar{s}$ is greater than one, then the slope of $\underline{r}$ will also be greater than one.

We can use equations (11) and (12) to iteratively construct upper and lower bounds on the set of iteratively undominated reporting strategies. We now show that $\underline{r}(1)$ increases monotonically to 1. Recall that in order for our construction of $\bar{r}(\cdot)$ to be valid, we required that the slope of $\underline{r}(\cdot)$ to be greater than $\frac{a}{a+n-1}$. Since this was satisfied by the initial lower bound, it will be satisfied by every subsequent lower bound provided the slopes increase along the sequence. Since each bound is linear, it suffices that the intercept $\underline{r}(1)$ increases monotonically. Furthermore, since the slope of $\underline{r}(\cdot)$ is less than one, it will follow that $\underline{r}(\cdot)$ increases monotonically to the identity mapping. Plugging in the identity mapping to equation 11 shows that the limit of $\bar{r}(\cdot)$ must also be the identity. Thus showing that $\underline{r}(1)$ increases to 1 will complete the proof.

Consider any value $0 < \underline{r}(1) < 1$. By linearity, the inverse is defined by $\underline{s}(r) = r/\underline{r}(1)$. Using equation (11), we can solve for $\bar{s}(r)$:

$$\bar{s}(r) = r\left[\frac{a+n-1-\frac{n-1}{\underline{r}(1)}}{a}\right]$$

Plugging in to equation (12) and solving for $\underline{r}'(1)$ (after some manipulation):

$$\underline{r}'(1) = \frac{a^2}{a^2 + \left[\frac{1-\underline{r}(1)}{\underline{r}(1)}\right](n-1)^2}$$

Since $\underline{r}(1) < 1$ and $a > n-1$, $\underline{r}'(1) > \underline{r}(1)$. This finishes the proof of the "if" part.

We now prove that the generalized VCG mechanism, for any arbitrary f_i's, is not dominance solvable if $a \leq n-1$. Consider the subset of reporting strategy profiles $\beta := \prod \beta_i$ where $\beta_i = \{r_i \in R_i : r_i(s_i) = \alpha + \frac{a}{a+n-1} s_i, \alpha \in [0, \frac{n-1}{a+n-1}]\}$. We will show that for any reporting strategy $r_i \in \beta_i$ and alternative reporting strategy $r_i' \in B_i$, there exists some reporting strategy $r_{-i} \in \beta_{-i}$ against which r_i does strictly better than r_i' does. Hence β must be a subset of any dominance solution, and hence there is no efficient dominance solution.

Fix any reporting strategy $r_i \in \beta_i$ and alternative reporting strategy $r_i' \in B_i$. Since $r_i' \neq r_i$, there exists $\hat{s}_i \in S_i$ such that $r_i'(\hat{s}_i) \neq r_i(\hat{s}_i)$. Suppose $r_i(\hat{s}_i) < r < r_i'(\hat{s}_i)$. Consider $r_j \in \beta_j$, $j \neq i$, such that $r_j(s_j) = \alpha + \frac{a}{a+n-1} s_j$, where $\alpha = \min\{r, \frac{n-1}{a+n-1}\}$. At the state of the world $s_i = \hat{s}_i$ and $\forall j \neq i$, $s_j = \frac{a+n-1}{a}(r - \alpha)$, employing reporting strategy r_i results in bidder i losing the object and getting net payoff $-f_i(r_{-j})$. Whereas employing reporting strategy r_i' results in bidder i winning the object and getting net payoff $a\hat{s}_i + (n-1)\frac{a+n-1}{a}(r-\alpha) - (a+n-1)r - f_i(r_{-i})$. If $r > \frac{n-1}{a+n-1}$, then $\alpha = \frac{n-1}{a+n-1}$, and the difference will be

$$
\begin{aligned}
& a\hat{s}_i + (n-1)\frac{a+n-1}{a}\left(r - \frac{n-1}{a+n-1}\right) - (a+n-1)r \\
= \ & a\hat{s}_i + (a+n-1)\frac{n-1-a}{a} r - \frac{(n-1)^2}{a} \\
< \ & a\hat{s}_i + (a+n-1)\frac{n-1-a}{a} - \frac{(n-1)^2}{a} \\
= \ & a\hat{s}_i - a \\
\leq \ & 0,
\end{aligned}
$$

where the strict inequality follows from $r < r_i'(\hat{s}_i) \leq 1$. If $r \leq \frac{n-1}{a+n-1}$, then $\alpha = r$, and the difference will be

$$
\begin{aligned}
& a\hat{s}_i - (a+n-1)r \\
< \ & a\hat{s}_i - (a+n-1)r_i(\hat{s}_i) \\
\leq \ & a\hat{s}_i - (a+n-1)\frac{a}{a+n-1}\hat{s}_i \\
= \ & 0,
\end{aligned}
$$

where the strict inequality follows from the definition of r. So employing reporting strategy r_i' unambiguously results in lower net payoff at the state of the world $s_i = \hat{s}_i$ and $\forall j \neq i$, $s_j = \frac{a+n-1}{a}(r-\alpha)$. The case where $r_i'(\hat{s}_i) < r < r_i(\hat{s}_i)$ is handled symmetrically. Combining the two cases we conclude that there always exists some reporting strategies $r_{-i} \in \beta_{-i}$ against which r_i does strictly better than r_i' does. This completes our proof. □

Let's summarize what we have and what we have *not* proved in this section. We have shown that, in the symmetric linear setting with $n \geq 3$ bidders, the generalized VCG mechanism is dominance solvable if and only if $a > n-1$. Since the generalized VCG mechanism is already the unique direct mechanism that implements the efficient allocation in ex post equilibrium, it is hopeless to construct other direct mechanisms that ex post dominance implement the efficient allocation when $a \leq n-1$. We summarize all these in the following corollary.

Corollary 5.2. *In the symmetric linear setting with $n \geq 3$ bidders, there exists a direct mechanism that ex post dominance implements the efficient allocation if and only if $a > n-1$.*

However, we have *not* proved that $a > n-1$ is necessary for ex post dominance implementation of the efficient allocation. The reason is that we have only looked at direct mechanisms. For most other equilibrium concepts, it suffices to look at direct mechanisms in order to obtain necessary conditions for implementation. This is because the revelation principle holds for most other equilibrium concepts. Unfortunately, the revelation principle breaks down for dominance solvable mechanisms. Since this is an observation that should be of independent interest, we shall discuss it in a separate section.

6. FAILURE OF THE REVELATION PRINCIPLE

We shall give an example in this section to demonstrate how the revelation principle fails to hold for dominance solution. Our example may not be the simplest one one can conceive, and makes use of a rather rich set of social alternatives. But it makes a point which we believe has not been addressed before in the literature. In the implementation literature, the revelation principle holds for most of the equilibrium concepts as long as we do not require unique implementation. However, when it comes to ex post dominance implementation (unique or not), the revelation principle breaks down.

Consider a situation with seven social alternatives (x, y, w_A, w_B, z_A, z_B, and o), two players (1 and 2), and each player having two types ($s_i = i_A, i_B$, $i = 1,2$). The players have quasi-linear utilities, with the corresponding valuation functions being summarized by Table 1. Table 1 also depicts the efficient rule, f, that the mechanism designer wants to implement.

$(s_1,s_2) =$	$(1_A,2_A)$	$(1_A,2_B)$	$(1_B,2_A)$	$(1_B,2_B)$
$v_1(x) = v_2(x) =$	2	0	0	2
$v_1(y) = v_2(y) =$	0	2	2	0
$v_1(w_A) =$	0	3	0	3
$v_1(w_B) =$	3	0	3	0
$v_1(z_A) = v_1(z_B) = v_1(o) =$	0	0	0	0
$v_2(z_A) =$	0	0	3	3
$v_2(z_B) =$	3	3	0	0
$v_2(w_A) = v_2(w_B) = v_2(o) =$	0	0	0	0
$f(s_1,s_2) =$	x	y	y	x

Table 1: Players' valuations and the efficient rule.

We first show that no dominance solvable direct mechanisms can (truthfully) implement f. Suppose an implementing direct mechanism exists. Such a direct mechanism will be fully characterized by a pair of transfer functions, $t_1(s_1,s_2)$ and $t_2(s_1,s_2)$.

If truth-telling is the dominance solution, there must be some untruthful strategy that is dominated for some player, say player 1. An untruthful strategy is dominated for player 1 only if there is some type, say 1_A for player 1 such that telling the truth (reporting A), is at least as good as lying (reporting B), for every type and report of player 2. This yields the following set of inequalities, with at least one strict.

$$v_1(f(1_A,2_A)|1_A,2_A)+t_1(1_A,2_A) \geq v_1(f(1_B,2_A)|1_A,2_A)+t_1(1_B,2_A) \tag{13}$$
$$v_1(f(1_A,2_B)|1_A,2_A)+t_1(1_A,2_B) \geq v_1(f(1_B,2_B)|1_A,2_A)+t_1(1_B,2_B) \tag{14}$$
$$v_1(f(1_A,2_A)|1_A,2_B)+t_1(1_A,2_A) \geq v_1(f(1_B,2_A)|1_A,2_B)+t_1(1_B,2_A) \tag{15}$$
$$v_1(f(1_A,2_B)|1_A,2_B)+t_1(1_A,2_B) \geq v_1(f(1_B,2_B)|1_A,2_B)+t_1(1_B,2_B). \tag{16}$$

Using the valuations from table 1, (14) and (15) reduce to

$$t_1(1_A,2_A) \geq 2+t_1(1_B,2_A) \tag{17}$$
$$t_1(1_A,2_B) \geq 2+t_1(1_B,2_B). \tag{18}$$

Since truth-telling is a dominance solution, it is also an ex post equilibrium in this finite game. Therefore truthful reporting should be a best-response for 1 to truthful reporting by 2 for every type profile, in particular whenever player

1 is of type 1_B:

$$v_1(f(1_B,2_A)|1_B,2_A)+t_1(1_B,2_A) \geq v_1(f(1_A,2_A)|1_B,2_A)+t_1(1_A,2_A)$$
$$v_1(f(1_B,2_B)|1_B,2_B)+t_1(1_B,2_B) \geq v_1(f(1_A,2_B)|1_B,2_B)+t_1(1_A,2_B).$$

This can be rewritten as

$$t_1(1_A,2_A) \leq 2+t_1(1_B,2_A) \tag{19}$$
$$t_1(1_A,2_B) \leq 2+t_1(1_B,2_B). \tag{20}$$

In view of (17)-(18), inequalities (19) and (20) are satisfied with equality implying that player 1 is indifferent between reporting A or B when type 1_B and 2 tells the truth.

Notice that (17) and (18) also imply

$$2+t_1(1_A,2_A) \geq t_1(1_B,2_A) \tag{21}$$
$$2+t_1(1_A,2_B) \geq t_1(1_B,2_B) \tag{22}$$

so that

$$v_1(f(1_A,2_A \mid 1_B,2_B)+t_1(1_A,2_A) \geq t_1(1_B,2_A) \tag{23}$$
$$v_1(f(1_A,2_B \mid 1_B,2_A)+t_1(1_A,2_B) \geq t_1(1_B,2_B) \tag{24}$$

i.e. 1 is willing to report A when he is type 1_B and 2 is lying.

We have shown that 1 is willing to report A for every type profile and for every behavior of player 2. Moreover, at least one of the preferences is strict (in (13)-(16)). Therefore, the strategy of reporting A independent of type is an ex post weakly dominant strategy for 1 and hence could never be deleted. But then truthful reporting could not be the dominance solution.

Now we demonstrate that f can be ex post dominance implemented with an *indirect* mechanism, and hence the revelation principle fails. The indirect mechanism has message space $\{i_A,i_B,i_C\}$ for each player $i=1,2$ (i_C can be interpreted as a "fictitious type" that is augmented to the message space of the direct mechanism), and transfer functions identical to zero. The outcome function is depicted in Figure 3.

The process of iterative elimination of dominated strategies goes as follows. In the first round, for any player $i=1,2$, when his true type is i_A, reporting i_B is dominated by reporting i_C; similarly, when his true type is i_B, reporting

	"2_A"	"2_B"	"2_C"
"1_A"	0	z_B	z_A
"1_B"	w_B	x	y
"1_C"	w_A	y	x

Figure 3: A dominance solvable indirect mechanism.

i_A is dominated by reporting i_C. In the second round, for any player $i = 1,2$, reporting i_C is dominated no matter what his true type is. So after two rounds of elimination, only truth-telling survives. We urge the reader to do the straightforward verification that truth-telling is an ex post equilibrium both for this indirect mechanism and for its corresponding direct mechanism (i.e., the direct mechanism derived by eliminating messages not used on the equilibrium path).

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Journal of Mechanism and Institution Design
ISSN: 2399-844X (Print), 2399-8458 (Online)
DOI:10.22574/jmid.2019.11.002

A NEW EVALUATION CRITERION FOR ALLOCATION MECHANISMS WITH APPLICATION TO VEHICLE LICENSE ALLOCATIONS IN CHINA

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ABSTRACT

In this paper, we propose an equality measure for allocation mechanisms with budget constraints to describe the difference in object obtaining opportunities among buyers with different budget ranks. We evaluate allocation mechanisms not only from the perspective of efficiency and revenue, but also with the criterion of equality. As an application of this new evaluation criterion – the equality measure, we study the vehicle license allocation problem in China, introduce a class of hybrid auction-lottery mechanisms, and evaluate China's vehicle license allocation in a unified framework from the criteria of efficiency, equality, and revenue.

We are grateful for helpful comments from Jinhui Bai, Atsushi Kajii, Jaimie Lien, Liguo Lin, Shigehiro Serizawa, Qianfeng Tang, Quan Wen, Zaifu Yang, Ning Yu, Yongchao Zhang, Jie Zheng, and Yu Zhou. Any errors are our own. The authors declare there are no conflicts of interest. Rong is financially supported by the Key Program of the National Nature Science Foundation of China (Grant No.U1601218) and the Major Program of National Social Science Foundation of China (Grant No.18ZDA041). Sun acknowledges financial support from China's "Cultural Celebrities and Four Batches of Talents" program. Wang is supported by the Youth Fund Project of Humanities and Social Sciences Research of Ministry of Education (Grant No.19YJC790129).

Keywords: Equality under budget constraints, hybrid mechanism, vehicle license allocation.

JEL Classification Numbers: D44, D47.

1. INTRODUCTION

In auction theory and market design, allocation mechanisms are usually evaluated solely by the criteria of efficiency and revenue. Nevertheless, in the allocation of public resources, equality should not be neglected for the sake of efficiency and revenue, since the conflict between efficiency and equality, as Okun (1975) points out, is an important tradeoff in economics and resource allocation. Indeed, in the practice of public resource allocation, equality is often taken into account, and non-price and hybrid mechanisms are often applied for the sake of equality. For example, public school admission and public rental housing are usually allocated through lotteries in China. Public health care is allocated by queues in countries such as UK and Canada. The rights of passage through the Panama Canal, hunting permits in several U.S. states, and U.S. immigration visas are allocated through a combination of price and non-price mechanisms. To sum up, the importance of equality in both economics and the practice of public resource allocation necessitates that we evaluate allocation mechanisms by the criterion of equality, in addition to efficiency and revenue.

Public opinion believes that, for some publicly-provided goods, equality of allocation requires that people with different wealth levels should have more or less equal chance of obtaining these goods.[1] People also believe that lottery, which allocates goods to different people with absolutely equal chance, is the most equal mechanism. Based on these common understandings of equality, we think that the equality of an allocation mechanism should reflect the difference in object obtaining opportunities among buyers with different wealth levels, and we shall propose one method to measure this difference. In this paper, we provide a class of general incentive compatible (IC) random direct mechanisms with budget constraints to describe the allocation of

[1] For example, Williams (2010) argues that "the notion of equality of opportunity...[is] that a limited good shall in fact be allocated on grounds which do not *a priori* exclude any section of those that desire it", and he believes that allocating some goods on grounds of wealth constitutes such an *a priori* exclusion. (Williams (2010), pp.243-244.)

publicly provided homogeneous goods, and propose a proper equality measure to evaluate such mechanisms. More specifically, for any IC random direct mechanism, we compute the expected object obtaining probabilities of buyers with different wealth levels (budgets), draw a Lorenz curve with these expected probabilities, and define a formal equality measure that is analogous to the Gini coefficient.[2] By proposing such a new evaluation criterion, we fill a gap in literature and make a contribution to auction theory.

It is worth emphasizing that we use a budget-constraint model to present our equality measure for two main reasons. As mentioned above, we believe that the equality of an allocation mechanism reflects the difference in object obtaining opportunities among buyers with different wealth levels. In this paper, we use a budget-constraint model to represent the heterogeneity in buyers' wealth levels. In addition, if the publicly-provided goods are large durable goods, the quasi-linear utility assumption is no longer valid for the allocation of those goods. In literature, there are usually two approaches that deal with large durable goods: budget constraints, or non-linear utility. In this study, we incorporate budget constraints into our analysis.

After establishing a proper equality measure for public resource allocation mechanisms, we proceed to an application of our equality measure, i.e., vehicle license allocation in China, because it is representative of allocation of publicly-provided goods, and is also important in China. To evaluate China's major vehicle license allocation mechanisms in a unified framework, and to provide new insights for improving license allocation, we propose a class of hybrid auction-lottery mechanisms. To conveniently compute the characteristics (efficiency, equality, and revenue) of the hybrid mechanisms, we further provide a continuum-mass hybrid mechanism for each discrete hybrid mechanism, present the formulas for its characteristics, and discuss the false-name bidding proof condition under the continuum-mass hybrid mechanism. Furthermore, to give a benchmark to compare with the hybrid mechanisms, we propose a probability allocation mechanism which relaxes the requirement of ex-post individual rationality. Finally, using simulation and numerical computation, we verify the robustness of approximating the characteristics of discrete hybrid mechanisms from those of continuum-mass hybrid mechanisms, depict the set of attainable characteristics of the hybrid mechanisms, and demonstrate that there is considerable room for improvement for some license allocation

[2] Of course, there may exist different methods to measure this difference, so the equality measure is not unique.

mechanisms in China.

Our study is connected with several strands of literature, which covers equality in public resource allocation, auctions with budget constraints, hybrid auction-lottery mechanisms, and vehicle license allocation. We will introduce them briefly.

Equality is an important issue in literature on public resource allocation. For example, Kahneman et al. (1986) discuss public standards of fairness for market allocations and indicate that some market anomalies can be explained by introducing fairness or equality. Taylor et al. (2003) argue that "lotteries are usually employed to resolve allocation problems in order to reflect a spirit of fairness and equality" (p.1316). Evans et al. (2009) observe that the choice between market and non-market mechanisms reflects the trade-off between efficiency and equity. For vehicle license allocations, Chen & Zhao (2013) demonstrate that in their survey, most respondents held negative attitudes toward the equity of the Shanghai auction. Although these articles mention the importance of equality, they neither clarify the meaning of equality, nor provide a formal equality measure of allocation mechanisms. Thus, they cannot compare different mechanisms in terms of equality. In fact, Taylor et al. (2003) and Evans et al. (2009) do not compare the equality of different mechanisms. Dworczak et al. (2019) define an equality measure and design optimal mechanism under the setting of non-linear utility, however their approach is not connected with the rich literature on mechanism design with budget constraints, and is unintuitive to apply. In this paper, we clarify the meaning of equality, and provide a formal equality measure of allocation mechanisms with budget constraints.

Auctions with budget constraints have been widely discussed in the literature. Laffont & Roberts (1996) examine an optimal sealed-bid single-item auction with budget constrained bidders. Che & Gale (1998) analyze bidding strategies, as well as efficiency and revenue in standard single-object auctions with private budget constraints, and find that first-price auctions yield higher revenue and social surplus than second-price auctions when buyers face absolute budget constraints. Maskin (2000) investigates second-price single-item auctions and all-pay single-item auctions under financial constraints. Che & Gale (2000) discuss the optimal mechanism of selling an object to a buyer, and Pai & Vohra (2014) further characterize optimal single-unit budget-constrained auctions. Talman & Yang (2015) propose an efficient dynamic auction for a market where there are multiple heterogenous items for sale and every bidder demands at most one item but faces a budget constraint. They show that their auction always finds a core allocation. Laan van der & Yang (2016) study a similar market and develop an ascending auction which locates a constrained equilibrium. Li (2017) provides a two-stage surplus-maximizing mechanism including cash subsidies, lotteries, and resale tax. In this paper, to apply the equality to broader classes of allocation mechanisms, we adopt a discrete multi-unit allocation model to discuss allocation mechanism with budget constraints.

Another related strand of literature focuses on hybrid auction-lottery mechanisms. Evans et al. (2009), in a paper similar to our study, examine the hybrid mechanisms with an auction implemented before a lottery and discuss buyers' bidding strategies in such mechanisms. Condorelli (2013) reveals that if buyers' values and willingness to pay do not align, a hybrid mechanism may be efficiency-optimal. Che et al. (2013) confirm this result and prove that an efficiency-optimal mechanism with budget constraints involves an in-kind subsidy and a cash incentive for discouraging low-valuation buyers from claiming the good. Our study adds a common reserve price to the hybrid mechanisms to raise both efficiency and revenue.

Several studies discuss vehicle license allocation in practice. Liao & Holt (2013) examine the modification of Shanghai license auction in 2008 and indicate through experiments that this modification, which aims to curb revenue, will result in loss of efficiency. Huang & Wen (2019) present buyers' bidding strategies under the Guangzhou mechanism. To our knowledge, no studies have yet considered the social planner's overall objectives in designing a vehicle license allocation mechanism and examined the existing license allocation mechanisms in a unified framework.

The remainder of the study is organized as follows. Section 2 describes the basic environment, proposes a class of random direct mechanisms to generalize public resource allocation mechanisms with budget constraints, and defines an equality measure for evaluating such mechanisms in terms of equality, in addition to efficiency and revenue. Section 3, 4, 5 and 6 study vehicle license allocation in China as an application of the equality measure. Specifically, Section 3 introduces vehicle license allocation in China, and proposes a class of hybrid mechanisms. Section 4 introduces the continuum-mass hybrid mechanisms, presents the formulas of its characteristics, and discusses buyers' incentives for false-name bidding. Section 5 presents the probability allocation mechanism as a benchmark mechanism. Section 6 employs numerical analysis to present the attainable characteristics of different mechanisms with figures, and evaluate mechanisms in terms of these characteristics. All proofs are provided in Appendix A.

2. EQUALITY MEASURE FOR ALLOCATION MECHANISMS

In this section, we shall propose an equality measure for allocation mechanisms. We first present the basic model: a multi-unit auction model with budget constraints. To characterize allocation mechanisms with budget constraints, we define a class of IC random direct mechanisms. Under such an IC random direct mechanism framework, we then provide a new evaluation criterion for allocation mechanisms besides the canonical criteria of efficiency and revenue.

2.1. Basic model

A social planner wishes to allocate m units of publicly-provided goods to n buyers. Every buyer $i \in \mathcal{N} = \{1, 2, \cdots, n\}$ is assumed to have unit demand, holds a value v_i, and is subject to a budget of w_i. Buyer i's private type $x_i = (v_i, w_i)$ is drawn from $\mathcal{X}_i = [0, \bar{v}] \times [0, \bar{w}]$ (here, $\bar{v}$ and $\bar{w}$ may be $+\infty$) according to a commonly known joint distribution function $\Phi(v, w)$ with density function $\phi(v, w)$. We assume that $\Phi(v, w)$ is strictly increasing in both v and w. Buyers' types are mutually independent. Each buyer knows her own type but not others' types. Every buyer $i \in \mathcal{N}$ has a utility function that is quasi-linear up to her budget constraint,

$$u_i(q_i, p_i, v_i, w_i) = \begin{cases} q_i v_i - p_i, & \text{if } p_i \leq w_i, \\ -\infty, & \text{if } p_i > w_i, \end{cases}$$

where q_i is buyer i's probability of obtaining an object and p_i is her required payment.

First, we introduce some notation. Let $\mathcal{X} = \times_{i=1}^{n} \mathcal{X}_i$ and $\mathcal{X}_{-i} = \times_{j \neq i} \mathcal{X}_j$ denote the space of all buyers' types, and the space of types of all buyers excluding i, respectively. Then, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{X}$ represents a profile of all buyers' types, and $\boldsymbol{x}_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathcal{X}_{-i}$ represents a type profile of all buyers but i. Write $\psi(\boldsymbol{x}) = \Pi_{i=1}^{n} \phi(x_i)$ as the joint density function of all buyers' types, and $\psi_{-i}(\boldsymbol{x}_{-i}) = \Pi_{j \neq i} \phi(x_j)$ as the joint density function of the types of all buyers but i. Let

$$F(v)=\int_0^v\int_0^{\bar{w}}\phi(t,w)dwdt \quad \text{and} \quad G(w)=\int_0^{\bar{v}}\int_0^{w}\phi(v,t)dtdv$$

denote the marginal distribution functions of a buyer's value and budget, respectively. The marginal density functions of a buyer's value and budget are

$$f(v)=\int_0^{\bar{w}}\phi(v,w)dw \quad \text{and} \quad g(w)=\int_0^{\bar{v}}\phi(v,w)dv,$$

respectively. Write $F(v|w)=\int_0^v \frac{\phi(t,w)}{g(w)}dt$ as the conditional distribution function of a buyer's value when her budget is w. We impose the following assumption on the distribution of buyers' types.

Assumption 1. *For any pair $\{w,w'\}$ with $w\leq w'$, the conditional distribution function $F(v|w')$ first-order stochastically dominates $F(v|w)$. That is, $F(v|w')\leq F(v|w)$ for all $v\in[0,\bar{v}]$.*

Assumption 1 is standard in the literature, and it implies that a buyer with a higher budget is more likely to have a higher value.

2.2. Random direct mechanism

By the Revelation Principle, we can restrict attention to incentive compatible direct mechanisms (abbreviated as IC direct mechanisms) in which every buyer $i\in\mathcal{N}$ prefers to report her true type $x_i=(v_i,w_i)\in\mathcal{X}_i$. Usually, a direct mechanism is defined by the interim assignment and the expected payment rules. However, in the presence of budget constraints, interim assignment and payment rules are not adequate for characterizing a direct mechanism,[3] because buyers' incentive properties may depend on the ex-post assignment and payment. Therefore, in this study, we shall consider the random direct mechanisms, in which interim assignments and payments are implemented by random assignment and payment rules that never make a buyer pay over her budget report.

Before defining random direct mechanisms, we first define the ex-post allocation of an allocation mechanism. Each ex-post assignment of objects

[3] To better appreciate the necessity of introducing random direct mechanisms, one can refer to the probability allocation mechanism presented in Section 5.

can be described as a vector $\pi \in \mathbb{Z}^n$ with each $\pi_i = 0$ or 1 and $\sum_{i \in \mathcal{N}} \pi_i \leq m$, where $\pi_i = 1$ (or, 0) represents that buyer i obtains (or does not obtain) an object. Let $\Pi = \{\pi \in \mathbb{Z}^n | \pi_i = 0, 1,\text{ and } \sum_{i \in \mathcal{N}} \pi_i \leq m\}$ denote the set of all possible ex-post assignments. Then, the set of all possible ex post allocations (allocation = assignment + payment) can be written as $\Omega = \Pi \times [0, \bar{w}]^n$. Let $\mathcal{F}(\Pi)$ and $\mathcal{F}(\Omega)$ be the spaces of all random vectors defined on Π and Ω, respectively.

We define a *random direct mechanism* for public resource allocation as a mapping

$$\mathbf{\Gamma} = (\Gamma_{11}, \Gamma_{21}, \ldots, \Gamma_{n1}, \Gamma_{12}, \Gamma_{22}, \ldots, \Gamma_{n2}) \ : \ \mathcal{X} \ \rightarrow \ \mathcal{F}(\Omega)$$

such that $\text{Prob}\{\Gamma_{i2}(\hat{\boldsymbol{x}}) \leq \hat{w}_i\} = 1$ for all $i \in \mathcal{N}$ and $\hat{\boldsymbol{x}} \in \mathcal{X}$.

For a report profile $\hat{\boldsymbol{x}}$, $\mathbf{\Gamma}(\hat{\boldsymbol{x}}) = (\Gamma_{11}(\hat{\boldsymbol{x}}), \ldots, \Gamma_{n1}(\hat{\boldsymbol{x}}), \Gamma_{12}(\hat{\boldsymbol{x}}), \ldots, \Gamma_{n2}(\hat{\boldsymbol{x}}))$ denotes a random allocation. The social planner assigns an object to each buyer i according to the random variable $\Gamma_{i1}(\hat{\boldsymbol{x}})$, and extracts payment from buyer i according to the random variable $\Gamma_{i2}(\hat{\boldsymbol{x}})$. Each realization $(\pi(\hat{\boldsymbol{x}}), p(\hat{\boldsymbol{x}}))$ of $\mathbf{\Gamma}(\hat{\boldsymbol{x}})$ denotes an ex-post outcome of the random allocation $\mathbf{\Gamma}(\hat{\boldsymbol{x}})$, where each buyer i gets $\pi_i(\hat{\boldsymbol{x}})$ object and pays $p_i(\hat{\boldsymbol{x}})$. The condition $\text{Prob}\{\Gamma_{i2}(\hat{\boldsymbol{x}}) \leq \hat{w}_i\} = 1$ ensures that no matter what other buyers report, each buyer i's ex-post payment can never exceed her reported budget. Note that these $2n$ random variables $\Gamma_{ij}(\hat{\boldsymbol{x}})$ $(i \in \mathcal{N}, j = 1, 2)$ may be inter-dependent.

For a given random direct mechanism $\mathbf{\Gamma}$ and an arbitrary $\hat{\boldsymbol{x}} \in \mathcal{X}$, let

$$Q_i(\hat{\boldsymbol{x}}) = E[\Gamma_{i1}(\hat{\boldsymbol{x}})] \quad \text{and} \quad M_i(\hat{\boldsymbol{x}}) = E[\Gamma_{i2}(\hat{\boldsymbol{x}})], \quad \text{for each } i \in \mathcal{N},$$

represent buyer i's expected probability of obtaining an object and expected payment, respectively. Thus, we obtain a normal interim direct mechanism

$$(\mathrm{Q}, \mathrm{M}) = (Q_1, \ldots, Q_n, M_1, \ldots, M_n) \ : \ \mathcal{X} \rightarrow \Delta \times [0, \bar{w}]^n,$$

where $\Delta = \left\{q \in \mathbb{R}^n_+ \mid q_i \in [0, 1], \sum_{i=1}^n q_i \leq m\right\}$. Henceforth, we refer to (Q, M) as the *associated direct mechanism* of $\mathbf{\Gamma}$. In addition, we say a random direct mechanism $\mathbf{\Gamma}$ is *standard* if its associated direct mechanism (Q, M) satisfies the following two properties.

1. **Anonymity** For any $i, j \in \mathcal{N}$, it holds that $Q_i(\bar{\boldsymbol{x}}) = Q_j(\hat{\boldsymbol{x}})$ and $M_i(\bar{\boldsymbol{x}}) = M_j(\hat{\boldsymbol{x}})$ for all $\bar{\boldsymbol{x}}, \hat{\boldsymbol{x}} \in \mathcal{X}$ satisfying $\bar{x}_i = \hat{x}_j$, $\bar{x}_j = \hat{x}_i$, and $\bar{x}_l = \hat{x}_l$ for all $l \in \mathcal{N} \setminus \{i, j\}$.

2. **Monotonicity** For any $i \in \mathcal{N}$, it holds that $Q_i(\hat{x}_i, \hat{\boldsymbol{x}}_{-i}) \geq Q_i(\hat{x}'_i, \hat{\boldsymbol{x}}_{-i})$ for all $\hat{\boldsymbol{x}}_{-i} \in \mathcal{X}_{-i}$, $\hat{x}_i$, and $\hat{x}'_i \in \mathcal{X}_i$ such that $\hat{x}_i \geq \hat{x}'_i$.[4]

We just consider standard random direct mechanisms in this study, and hence we shall omit the term "standard" without confusion.

For each buyer i and any type $\hat{x}_i \in \mathcal{X}_i$, let

$$\begin{aligned} q_i(\hat{x}_i) &= \int_{\mathcal{X}_{-i}} Q_i(\hat{x}_i, \boldsymbol{x}_{-i}) \psi_{-i}(\boldsymbol{x}_{-i}) d\boldsymbol{x}_{-i} \\ m_i(\hat{x}_i) &= \int_{\mathcal{X}_{-i}} M_i(\hat{x}_i, \boldsymbol{x}_{-i}) \psi_{-i}(\boldsymbol{x}_{-i}) d\boldsymbol{x}_{-i} \end{aligned} \tag{2.1}$$

denote her expected probability of winning a object and her expected payment when she reports $\hat{x}_i$ and all other buyers report their true types. By the anonymity of a standard mechanism and the identical distribution of buyers' types, we see that all buyers are symmetric from an ex-ante perspective. Therefore, we use $q(v, w)$ to represent $q_i(v, w)$ and use $m(v, w)$ to represent $m_i(v, w)$ for any $i \in \mathcal{N}$ and $(v, w) \in \mathcal{X}_i$.

We say a random direct mechanism $\boldsymbol{\Gamma}$ is *interim individually rational* if, for all $i \in \mathcal{N}$ and $\hat{\boldsymbol{x}}_{-i} \in \mathcal{X}_{-i}$, the following is satisfied:

$$u_i(Q_i(x_i, \hat{\boldsymbol{x}}_{-i}), M_i(x_i, \hat{\boldsymbol{x}}_{-i}), x_i) = Q_i(x_i, \hat{\boldsymbol{x}}_{-i}) v_i - M_i(x_i, \hat{\boldsymbol{x}}_{-i}) \geq 0,$$

for all $x_i \in \mathcal{X}_i$. We also say a random direct mechanism $\boldsymbol{\Gamma}$ is *ex-post individually rational* if, for all $i \in \mathcal{N}$, $x_i \in \mathcal{X}_i$ and $\hat{\boldsymbol{x}}_{-i} \in \mathcal{X}_{-i}$, it satisfies that

$$u_i(\pi_i(x_i, \hat{\boldsymbol{x}}_{-i}), p_i(x_i, \hat{\boldsymbol{x}}_{-i}), x_i) = \pi_i(x_i, \hat{\boldsymbol{x}}_{-i}) v_i - p_i(x_i, \hat{\boldsymbol{x}}_{-i}) \geq 0,$$

for any realization $(\pi_i(x_i, \hat{\boldsymbol{x}}_{-i}), p_i(x_i, \hat{\boldsymbol{x}}_{-i}))$ of $(\Gamma_{i1}(x_i, \hat{\boldsymbol{x}}_{-i}), \Gamma_{i2}(x_i, \hat{\boldsymbol{x}}_{-i}))$.

In addition, a random direct mechanism $\boldsymbol{\Gamma}$ is said to be *incentive compatible* if, for all $i \in \mathcal{N}$ and $x_i \in \mathcal{X}_i$, the following is satisfied:

$$\begin{aligned} & u_i(q(x_i), m(x_i), x_i) \\ = \; & q(x_i) v_i - m(x_i) \\ \geq \; & q(\hat{x}_i) v_i - m(\hat{x}_i) \\ = \; & u_i(q(\hat{x}_i), m(\hat{x}_i), x_i), \end{aligned}$$

[4] The property of "monotonicity" implies that given all other buyers' reports, a buyer's object obtaining probability is nondecreasing in her report type. In literature, a standard auction is an auction in which buyers who propose the highest bids always win the objects (Krishna, 2010), while in a standard mechanism defined here, a higher type reporting is always accompanied by a higher object obtaining probability.

for all $\hat{x}_i \in \mathcal{X}_i$ satisfying Prob$\{\Gamma_{i2}(\hat{x}_i, \boldsymbol{x}_{-i}) \leq w_i\} = 1$ for all $\boldsymbol{x}_{-i} \in \mathcal{X}_{-i}$. A random direct mechanism $\boldsymbol{\Gamma}$ is *weakly dominant strategy incentive compatible* if, for all $i \in \mathcal{N}$, $x_i \in \mathcal{X}_i$ and $\hat{\boldsymbol{x}}_{-i} \in \mathcal{X}_{-i}$, the following is satisfied:

$$\begin{aligned} & u_i(Q_i(x_i, \hat{\boldsymbol{x}}_{-i}), M_i(x_i, \hat{\boldsymbol{x}}_{-i}), x_i) \\ = & Q_i(x_i, \hat{\boldsymbol{x}}_{-i})v_i - M_i(x_i, \hat{\boldsymbol{x}}_{-i}) \\ \geq & Q_i(\hat{x}_i, \hat{\boldsymbol{x}}_{-i})v_i - M_i(\hat{x}_i, \hat{\boldsymbol{x}}_{-i}) \\ = & u_i(Q_i(\hat{x}_i, \hat{\boldsymbol{x}}_{-i}), M_i(\hat{x}_i, \hat{\boldsymbol{x}}_{-i}), x_i), \end{aligned}$$

for all $\hat{x}_i \in \mathcal{X}_i$ satisfying Prob$\{\Gamma_{i2}(\hat{x}_i, \hat{\boldsymbol{x}}_{-i}) \leq w_i\} = 1$.

It is obvious that ex-post individual rationality implies interim individual rationality, and weakly dominant strategy incentive compatibility implies incentive compatibility.

2.3. Characteristics of IC random direct mechanisms

When designing a public resource allocation mechanism, the social planner considers not only efficiency and revenue, but also equality. Therefore, for each IC random direct mechanism $\boldsymbol{\Gamma}$, we shall define its (ex-ante) characteristics of efficiency, revenue, and equality. Since the ex-ante features of a random direct mechanism $\boldsymbol{\Gamma}$ are usually determined by its associated (interim) direct mechanism (Q, M), we shall use (Q, M) to define the characteristics of $\boldsymbol{\Gamma}$.

Efficiency and revenue

Efficiency of a public resource allocation mechanism describes whether those buyers with higher values are more likely to obtain objects. Given an IC random direct mechanism $\boldsymbol{\Gamma}$, its efficiency is usually defined as the aggregate realized values and its revenue is defined as all buyers' expected payments to the social planner. For convenience, in the setting of multi-unit item allocations, we adopt the expected realized values per object as the measure of efficiency and take the expected payments per object as the measure of revenue.[5] Formally, we have the following definitions.

Definition 1. *The efficiency measure of an IC random direct mechanism* $\boldsymbol{\Gamma}$ *is defined as*

$$Ef(\boldsymbol{\Gamma}) = \frac{1}{m} \int_{\mathcal{X}} \sum_{i=1}^{n} Q_i(\boldsymbol{x}) v_i \, \psi(\boldsymbol{x}) d\boldsymbol{x}. \tag{2.2}$$

[5] Note that in the efficiency and revenue measures we divide the total amount of values and payments by the number of objects m no matter how many objects are eventually allocated.

Definition 2. *The revenue measure of an IC random direct mechanism* $\mathbf{\Gamma}$ *is defined as*

$$Re(\mathbf{\Gamma}) = \frac{1}{m}\int_{\mathcal{X}}\sum_{i=1}^{n} M_i(\boldsymbol{x})\psi(\boldsymbol{x})d\boldsymbol{x}. \tag{2.3}$$

Equality

The equality of public resource allocation is an inescapable issue. For each IC random direct mechanism $\mathbf{\Gamma}$, we shall define its equality measure to measure the difference in winning opportunities among buyers with different budget ranks under $\mathbf{\Gamma}$.

For a given IC random direct mechanism $\mathbf{\Gamma}$ and a profile $\boldsymbol{x} \in \mathcal{X}$, $\boldsymbol{w} = (w_1, w_1, \ldots, w_n)$ and $\mathrm{Q}(\boldsymbol{x}) = (Q_1(\boldsymbol{x}), Q_2(\boldsymbol{x}), \ldots, Q_n(\boldsymbol{x}))$ represent the budgets and the interim assignments of all buyers, respectively. After sorting the vector $\boldsymbol{w}$ from low to high, we obtain a permutation $\sigma : \mathcal{N} \to \mathcal{N}$ such that

$$w_{\sigma(1)} \leq w_{\sigma(2)} \leq \ldots \leq w_{\sigma(n)},$$

where ties are broken randomly. For each $j = 1, 2, \ldots, n$, let $Q_{(j)}(\boldsymbol{x}) \equiv Q_{\sigma(j)}(\boldsymbol{x})$ denote the probability that the buyer with the j-th lowest budget obtains an object at profile $\boldsymbol{x}$,[6] and let $\hat{\boldsymbol{Q}}(\boldsymbol{x}) = (Q_{(1)}(\boldsymbol{x}), Q_{(2)}(\boldsymbol{x}), \ldots, Q_{(n)}(\boldsymbol{x}))$. Then, $\hat{\boldsymbol{Q}}(\boldsymbol{x})$ denotes the vector of buyers' winning probability ranked by their budgets from low to high in profile $\boldsymbol{x}$. For each $j = 1, 2, \ldots, n$, we further define

$$\overline{Q}_{(j)} = \int_{\mathcal{X}} Q_{(j)}(\boldsymbol{x})\psi(\boldsymbol{x})d\boldsymbol{x} \tag{2.4}$$

as the average probability of obtaining an object for buyers with the j-th lowest budgets. In other words, $\overline{Q}_{(j)}$ is a buyer's expected probability of obtaining an object when she only knows that her budget is the j-th lowest among all buyers. We use

$$p(w) = \int_0^{\bar{v}} q(v, w)dF(v|w)$$

[6] For example, suppose $\boldsymbol{w} = (3, 7, 6, 4, 5)$ and $\mathrm{Q}(\boldsymbol{x}) = (0.15, 0.3, 0.25, 0.1, 0.2)$. Then we get a permutation $\sigma = (1, 4, 5, 3, 2)$ by sorting all buyers' budgets from low to high. Reorder $\mathrm{Q}(\boldsymbol{x})$ by the permutation σ, i.e., rank all buyers' winning probabilities by their budgets from low to high. We then get a new vector $(Q_{(1)}, Q_{(2)}, Q_{(3)}, Q_{(4)}, Q_{(5)}) = (Q_1, Q_4, Q_5, Q_3, Q_2) = (0.15, 0.1, 0.2, 0.25, 0.3)$.

to denote a buyer's expected probability of obtaining an object when her budget is w. Then $\overline{Q}_{(j)}$ can be rewritten as

$$\begin{aligned}\overline{Q}_{(j)} &= \int_0^{\bar{w}} p(w)dG_{(j)}(w)\\ &= \int_0^{\bar{w}}\int_0^{\bar{v}}\int_{\mathcal{X}_{-i}} Q_i((v,w),\boldsymbol{x}_{-i})\psi_{-i}(\boldsymbol{x}_{-i})d\boldsymbol{x}_{-i}dF(v|w)dG_{(j)}(w),\end{aligned} \tag{2.5}$$

where $G_{(j)}(w) = \sum_{i=j}^{n}\binom{n}{i}G^i(w)[1-G(w)]^{n-i}$ is the distribution function of the j-th lowest budget.

We thus obtain a vector $\overline{\boldsymbol{Q}} = (\overline{Q}_{(1)}, \overline{Q}_{(2)}, \ldots \overline{Q}_{(n)})$, which contains all information about the difference in object obtaining opportunities among buyers with different budget ranks. Based on $\overline{\boldsymbol{Q}}$, we can draw a Lorenz curve that describes the proportion of objects assigned to the poorest fraction of buyers. We further present the condition under which this Lorenz curve lies below the $45°$ line.

Lemma 1. *In an IC (standard) random direct mechanism* $\boldsymbol{\Gamma}$, *if Assumption 1 holds, then* $\overline{Q}_{(j)}$ *is nondecreasing in* j, *and the Lorenz curve lies below the* $45°$ *line.*

With this Lorenz curve, we can define an equality measure that describes the difference in winning opportunities among buyers with different budget ranks.

Definition 3. *The equality measure of an IC random direct mechanism* $\boldsymbol{\Gamma}$, *is defined as*

$$Eq(\boldsymbol{\Gamma}) = \frac{2}{n+1}\cdot\frac{1}{\sum_{j=1}^{n}\overline{Q}_{(j)}}\sum_{k=1}^{n}\sum_{j=1}^{k}\overline{Q}_{(j)}. \tag{2.6}$$

In Figure 1, A denotes the shaded area and B denotes the area in green. The Gini coefficient is $\frac{A}{A+B}$ and the equality measure is $\frac{B}{A+B}$. When n is sufficiently large, the upper edges of the shaded area and the green area become the $45°$ line and a smooth Lorenz curve, respectively. Thus, the Gini coefficient becomes $2A$, and the equality measure becomes $2B$.

It is worth noting that this equality measure just captures a bit of information of $\overline{\boldsymbol{Q}}$, and serves as a rough assessment of an allocation mechanism in

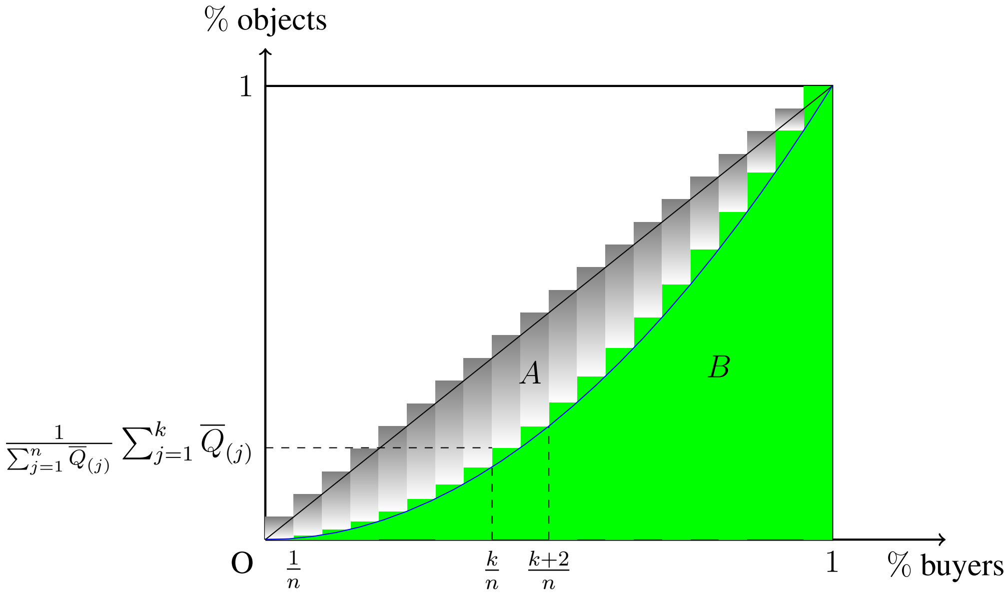


Figure 1: Lorenz curve of public resource allocation

terms of equality. To obtain more information about the impacts of allocation mechanisms on buyers from different social strata, one need to refer directly to the Lorenz curve.

For any IC random direct mechanism $\mathbf{\Gamma}$, we use

$$Ch(\mathbf{\Gamma}) = (Ef(\mathbf{\Gamma}), Eq(\mathbf{\Gamma}), Re(\mathbf{\Gamma}))$$

to denote its characteristics. Since concerning these characteristics, the social planner can be assumed to hold a social welfare function on the space of $Ch(\mathbf{\Gamma})$, and his objective is to choose an optimal allocation mechanism to maximize his social welfare function.

2.4. Continuum-mass IC random direct mechanism and its characteristics

In the above subsection, we present the characteristics of an IC direct mechanism, especially we define the equality measure for such a mechanism. Nevertheless, the equations (2.5) involved in the definition of the equality measure are rather complicated, making the computation of the equality measure difficult to be applied. Fortunately, if the sizes of buyers and objects are sufficiently large, we can describe the allocation problem with a continuum-mass model and simplify the computation of the equality measure. In this subsection, we will provide a continuum-mass version of the random direct mechanism, define its incentive properties, and present its characteristics of efficiency, equality, and revenue. We will first present the basic setting of the continuum-mass model.

A social planner wishes to allocate α mass of publicly-provided goods to a unit mass of buyers. Each buyer is assumed to have unit demand, holds a value v, and is subject to a budget of w. For any buyer, the distributions of her private type (v, w) and her utility function are identical to the settings in Subsection 2.1, and we will omit them here.

Continuum-mass random direct mechanism and its incentive properties
Let $\Omega' = \{0, 1\} \times [0, \bar{w}]$ represent the set of all possible ex-post allocations to a buyer, and let $\mathcal{F}(\Omega')$ be the space of all random vectors defined on Ω'. Following Che et al. (2013) and Richter (2019), we can define a continuum-mass random direct mechanism as a mapping

$$\mathbf{\Gamma} = (\Gamma_1, \Gamma_2) : [0, \bar{v}] \times [0, \bar{w}] \to \mathcal{F}(\Omega')$$

such that $\text{Prob}\{\Gamma_2(\hat{v}, \hat{w}) \leq \hat{w}\} = 1$ for all $(\hat{v}, \hat{w}) \in [0, \bar{v}] \times [0, \bar{w}]$.

For a given continuum-mass random direct mechanism $\mathbf{\Gamma}$ and an arbitrary report $(\hat{v}, \hat{w})$ of the buyer, let $(\pi(\hat{v}, \hat{w}), p(\hat{v}, \hat{w}))$ be a realization of $(\Gamma_1(\hat{v}, \hat{w}), \Gamma_1(\hat{v}, \hat{w}))$, and let

$$q(\hat{v}, \hat{w}) = E[\Gamma_1(\hat{v}, \hat{w})] \quad \text{and} \quad m(\hat{v}, \hat{w}) = E[\Gamma_2(\hat{v}, \hat{w})]$$

represent the buyer's probability of obtaining an object, and the expected payment she must take, respectively. We say a continuum-mass random direct mechanism $\mathbf{\Gamma}$ is *interim individually rational* if, for all $v \in [0, \bar{v}]$ and $w \in [0, \bar{w}]$, the following is satisfied:

$$vq(v, w) - m(v, w) \geq 0.$$

We also say $\mathbf{\Gamma}$ is *ex-post individually rational* if, for all $v \in [0, \bar{v}]$ and $w \in [0, \bar{w}]$, it satisfies that

$$\pi(v, w)v - p(v, w) \geq 0$$

for any realization $(\pi(v,w), p(v,w))$ of $(\Gamma_1(v,w), \Gamma_2(v,w))$, and $\mathbf{\Gamma}$ is said to be *incentive compatible* if, for all $v \in [0,\bar{v}]$ and $w \in [0,\bar{w}]$, the following is satisfied:

$$vq(v,w) - m(v,w) \geq vq(v',w') - m(v',w'),$$

for all (v',w') such that $\text{Prob}\{\Gamma_2(v',w') \leq w\} = 1$.

Characteristics of a continuum-mass IC random direct mechanism

We still adopt the expected realized values per unit of good as the measure of efficiency and consider the expected payments per unit of good as the measure of revenue. Formally, we have the efficiency measure and the revenue measure defined as:

$$Ef(\mathbf{\Gamma}) = \frac{1}{\alpha}\int_0^{\bar{w}}\int_0^{\bar{v}} q(v,w)v\phi(v,t)dvdt, \tag{2.7}$$

and

$$Re(\mathbf{\Gamma}) = \frac{1}{\alpha}\int_0^{\bar{w}}\int_0^{\bar{v}} m(v,w)\phi(v,t)dvdt. \tag{2.8}$$

If a buyer's budget is $w \in [0,\bar{w}]$, $p(w) = \int_0^{\bar{v}} q(v,w)dF(v|w)$ denoted her expected probability of obtaining an object, and

$$P(w) = \int_0^{w}\int_0^{\bar{v}} q(v,t)\phi(v,t)dvdt$$

denotes the cumulative mass of objects won by buyers with budgets no greater than w. Let $s \in [0,1]$ represent the mass of the buyers whose budgets are less than $w = G^{-1}(s)$. Thus, $L(s) = \frac{1}{\alpha}P(G^{-1}(s))$ denotes the fraction of objects allocated to the s fraction of buyers with the lowest budgets. The function $L(s) : [0,1] \to [0,1]$ have the properties that $L(0) = 0$, $L(1) = 1$ and $L'(s) = \frac{1}{\alpha}p(G^{-1}(s))$. Thus, when $q(v,w)$ is nondecreasing, it follows from Assumption 1 that $L'(s)$ is nondecreasing in s and the function $L(s)$ represents a well-defined Lorenz curve. Consequently, the equality measure is well-defined, and can be expressed as

$$\begin{aligned} Eq(\mathbf{\Gamma}) &= 2\int_0^1 L(s)\,ds = \frac{2}{\alpha}\int_0^{\bar{w}} P(w)dG(w) \\ &= \frac{2}{\alpha}\int_0^{\bar{w}}\int_0^{w}\int_0^{\bar{v}} q(v,t)\phi(v,t)dvdtdG(w). \end{aligned} \tag{2.9}$$

3. THE HYBRID MECHANISM FOR VEHICLE LICENSE ALLOCATION

From this section on, as an application of the new evaluation criterion — the equality measure, we shall study the vehicle license allocations in China.[7] Specifically, we will use a class of hybrid auction-lottery mechanisms to generalize several license allocation mechanisms in China in a unified framework, and to evaluate and improve upon these license allocation mechanisms in terms of equality, in addition to efficiency and revenue. In this section, we will first briefly introduce China's vehicle license allocation mechanisms in practice. Then, we provide a new class of discrete hybrid mechanisms to incorporate these mechanisms.

3.1. Vehicle license allocation in China

The rapid growth in private vehicle ownership in China has led to traffic jams and air pollution in big cities. To alleviate these problems, several cities have placed limits on vehicle license quota, and instituted different mechanisms to allocate the limited supply of vehicle licenses.

Since 2002, Shanghai has used a multi-unit, discriminatory price auction to allocate vehicle licenses. In July 2013, Shanghai modified the auction rule, and introduced a "warning price" that essentially serves as a price ceiling. The current Shanghai auction comprises two phases and each lasts for 30 minutes. In the first phase of the auction, each bidder submits a bid which cannot be higher than the warning price. In the second phase, up to two bid revisions are allowed for each bidder, and the revised bids are restricted to some interval (approximately 300 RMB) above the lowest bid at that moment. The narrow price window, the limited bidding chance, and the increasing lowest accepted bid together incentivize bidders to "snipe" in the last ten or twenty seconds of the auction. Therefore, the final trading prices are just slightly higher than the warning price. Indeed, during 2014, the warning price was fixed at 72,600 RMB and the auction prices remain between 73,000 and 74,000 RMB. The warning price is adjusted gradually in the following years, but during each

[7] Vehicle license allocation has been a heated topic in Shanghai since 2013. We have in several occasions, including in Rong & Sun (2015), briefly discussed vehicle license allocation in China and recommended the hybrid mechanism introduced below as a solution to Shanghai's vehicle license allocation. Formal definition of this mechanism and detailed discussion of its characteristics are given in this paper.

year, the warning price is roughly fixed. For example, in 2018, the warning price is set as 86,300 RMB, which is 5,400 RMB short of the warning price in 2017, and is far below the potential equilibrium price. Therefore, we state that the current Shanghai auction is a price ceiling mechanism. Nevertheless, the "price ceiling" of the Shanghai auction is implemented by buyers' sniping behavior, instead of an open and fair lottery. Therefore, many buyers, hoping to win licenses on time, either pay a large amount (about 20,000 RMB) to auction intermediaries with better internet connection, or try different bidding methods. The strategy complexity and redundant intermediaries remain severe problems faced by the current Shanghai auction.

Beijing and Guiyang have been implementing a vehicle license lottery since 2011 to ensure equal allocation. In 2018, the Beijing lottery featured a very low winning probability (roughly 0.05%) for participating buyers. Guangzhou, Tianjin, Hangzhou, and Shenzhen adopted a hybrid lottery-auction mechanism in 2012, 2013, 2014, and 2015 to allocate vehicle licenses. In the discussion that follows, we name this hybrid mechanism "the Guangzhou mechanism." In this mechanism, roughly half of the license quota are allocated by a discriminatory-price auction, and the remaining licenses are allocated by a lottery in which the winners need not pay anything for licenses. The auction and lottery are totally separate, namely, every buyer has to choose between entering the auction or lottery. Therefore, it is difficult for buyers to choose between these options.[8] In fact, the auction losers always regret not having chosen the lottery. Several other cities are also planning to institute a license quota. How to allocate the given license quota more efficiently and equally remains a challenge in China.

3.2. A general hybrid mechanism for vehicle license allocation in China

We propose a new class of hybrid mechanisms for three concerns: (1)to simplify buyers' bidding strategies, (2) to incorporate several China's vehicle license allocation mechanisms in a unified framework, (3) to provide the social planner with more flexible policy tools to improve license allocation. In our hybrid mechanisms, every buyer has the opportunity to enter both the auction and lottery. All buyers enter the auction first, with the auction losers then

[8] Buyers' bidding strategies in the Guangzhou mechanism may be rather complex. Huang & Wen (2019) study buyers' bidding behaviors under the Guangzhou mechanism, and provide an equilibrium bidding strategy.

entering the lottery. The lottery winners, unlike in the Guangzhou mechanism, are also required to pay a reserve price to the social planner. Formally, our hybrid mechanism is defined as follows:

Hybrid auction-lottery mechanism for vehicle licenses

Step 1 **Announcement** The social planner announces the total license quota m, the auction quota m_1, the lottery quota $m - m_1$, and a reserve price r.

Step 2 **Registration** All buyers decide whether to register for the mechanism. At the end of registration, the social planner announces the number of all registered buyers n_1.

Step 3 **Auction** All registered buyers submit their bids, which should be no less than r.[9] A buyer whose bid is among the m_1 highest bids wins the auction, obtains a license, and pays the $(m_1 + 1)$-th highest bid.[10]

Step 4 **Lottery** Those registered buyers who lose the auction enter the lottery in which $m - m_1$ licenses are allocated. Each lottery winner obtains a license and pays r.

Since the total license quota m is exogenously given and the social planner can only adjust r and m_1, we shall use $H(r, m_1)$ to denote such a hybrid mechanism. It is obvious that: (i) if $m_1 = 0$ and $r = 0$, then $H(r, m_1)$ becomes the Beijing mechanism; (ii) if $m_1 = m$ and $r = 0$, then $H(r, m_1)$ is roughly the Shanghai auction before July 2013. The current Shanghai mechanism with a warning price c is theoretically equivalent to the hybrid mechanism $H(c, 0)$ because in practice, the number of buyers willing to bid no less than the warning price c (denoted by m_c) is much more than the license quota m.[11]

For a hybrid mechanism $H(r, m_1)$, if $n_1 < m$, i.e., r is set so high that the number of registered buyers is less than the license quota, then $H(r, m_1)$

[9] If a registered buyer does not submit a bid, her bid is set as r by default.

[10] When the number of registered buyers n_1 is no greater than m_1, all registered buyers win the auction and pay r.

[11] If the warning price c is set so high that $m_c \leq m$, then the warning price does not take effect, and the current Shanghai mechanism with such a warning price reduces to the Shanghai auction before July 2013. Therefore, for any warning price c, the current Shanghai mechanism $M(c)$, essentially a price ceiling mechanism, can be induced from our hybrid mechanisms.

reduces to a posted price selling mechanism. Usually, for the allocation of scarce resource such as vehicle licenses, the reserve price is not set so high, thus registered buyers are more than objects allocated, i.e., $n_1 > m$. Let

$$\lambda = \begin{cases} \frac{m-m_1}{n_1-m_1}, & \text{if } n_1 > m, \\ 1, & \text{if } n_1 \leq m, \end{cases}$$

denote the probability of winning a license in the lottery. All buyers know λ because, according to the rules of $H(r, m_1)$, they can calculate it from n_1, m_1, and m.

In a hybrid mechanism $H(r, m_1)$, we say buyer i with type $x_i = (v_i, w_i)$ *bids sincerely* whenever

1. she registers for the mechanism, if and only if, $\min\{v_i, w_i\} \geq r$;
2. when she is a registered buyer, she bids $\min\{\, v_i - \lambda(v_i - r),\ w_i\,\}$.

We say a hybrid mechanism $H(r, m_1)$ is *ex-post individually rational* if, no matter how other buyers bid, each buyer i gets a non-negative ex-post utility when bidding sincerely. Clearly, by the rule of the hybrid mechanisms, each hybrid mechanism is ex-post individually rational. On sincere bidding strategy, we further have the following result.

Theorem 1. *Each hybrid mechanism $H(r, m_1)$ is ex-post individually rational. Moreover, in a hybrid mechanism $H(r, m_1)$, for every buyer i sincere bidding is a weakly dominant strategy.*

Theorem 1 implies that the hybrid mechanism is detail-free, namely, a buyer need not know the distributions of buyers' types when bidding. In addition, in a hybrid mechanism, every buyer is just required to report a bid instead of her whole type. In this sense, our hybrid mechanism is not a direct mechanism and is privacy preserving compared with direct mechanisms. Nevertheless, for every hybrid mechanism $H(r, m_1)$, by the Revelation Principle and Theorem 1, we can easily construct a relative IC random direct mechanism

Formally, we have

$$M(c) = \begin{cases} H(c, 0), & \text{if } m_c > m, \\ H(0, m), & \text{if } m_c \leq m. \end{cases}$$

$\mathbf{\Gamma}$ to implement the same outcome of $H(r, m_1)$.[12] By Theorem 1, we can further show this random direct mechanism $\mathbf{\Gamma}$ is ex-post individually rational and weakly dominant strategy incentive compatible. For a hybrid mechanism $H(r, m_1)$, we use the characteristics $Ch(\mathbf{\Gamma})$ of its relative IC random direct mechanism to represent its characteristics. In particular, we write the characteristics of $H(r, m_1)$ as $Ch(r, m_1) = (Ef(r, m_1), Eq(r, m_1), Re(r, m_1))$. The social planner chooses the proper hybrid mechanism $H(r, m_1)$ to maximize his social welfare function.

Finally, it is worth noting that Theorem 1 is based on an implicit assumption that every buyer bids with her true identity. In practice, some buyers may have incentives to register for the mechanism under false names and submit multiple bids to increase their expected payoff. Indeed, it is widely reported that some buyers participate in the Beijing mechanism with multiple identities to increase their probability of winning. We shall discuss this issue in Subsection 4.3.

[12] The relative IC random direct mechanism can be roughly defined as follows. Each buyer $i \in \mathcal{N}$ reports a type $\hat{x}_i = (\hat{v}_i, \hat{w}_i)$. The social planner first computes each buyer i's "bid" $b_i(\hat{v}_i, \hat{w}_i)$ by

$$b_i(\hat{v}_i, \hat{w}_i) = \begin{cases} \min\{\hat{v}_i - \lambda(\hat{v}_i - r),\ \hat{w}_i\}, & \text{if } \min\{\hat{v}_i, \hat{w}_i\} \geq r, \\ 0, & \text{if } \min\{\hat{v}_i, \hat{w}_i\} < r. \end{cases}$$

Let $b^{(m_1+1)}$ denote the $(m_1 + 1)$-th highest bid in all bids $\{b_i(\hat{x}_i) : i \in \mathcal{N}\}$. Thus, according to the Birkhoff-von Neumann theorem (Budish et al., 2013), there is a random assignment $\mathbf{\Gamma}_1(\hat{\boldsymbol{x}}) = (\Gamma_{11}(\hat{\boldsymbol{x}}), \ldots, \Gamma_{n1}(\hat{\boldsymbol{x}})) \in \mathcal{F}(\Pi)$ satisfying the following: (i) for each buyer i with $b_i(\hat{v}_i, \hat{w}_i) \geq b^{(m_1+1)}$, $\Gamma_{i1}(\hat{\boldsymbol{x}})$ reduces to a deterministic assignment 1; (ii) for each buyer i with $b_i(\hat{v}_i, \hat{w}_i) = 0$, $\Gamma_{i1}(\hat{\boldsymbol{x}})$ reduces to a deterministic assignment 0; and (iii) for each buyer i with $r \leq b_i(\hat{v}_i, \hat{w}_i) \leq b^{(m_1+1)}$, $\Gamma_{i1}(\hat{\boldsymbol{x}})$ satisfies $Q_i(\hat{\boldsymbol{x}}) \equiv E[\Gamma_{i1}(\hat{\boldsymbol{x}})] = \lambda$.

The random payment rule is constructed as follows: (i) for each buyer i with $b_i(\hat{v}_i, \hat{w}_i) \geq b^{(m_1+1)}$, $\Gamma_{i2}(\hat{\boldsymbol{x}})$ is a deterministic payment $b^{(m_1+1)}$; (ii) for each buyer i with $b_i(\hat{v}_i, \hat{w}_i) = 0$, $\Gamma_{i2}(\hat{\boldsymbol{x}})$ is a deterministic payment 0; and (iii) for each buyer i with $r \leq b_i(\hat{v}_i, \hat{w}_i) \leq b^{(m_1+1)}$, $\Gamma_{i2}(\hat{\boldsymbol{x}})$ is defined by

$$\Gamma_{i2}(\hat{\boldsymbol{x}}) = \begin{cases} r, & \text{if } \Gamma_{i1}(\hat{\boldsymbol{x}}) = 1, \\ 0, & \text{if } \Gamma_{i1}(\hat{\boldsymbol{x}}) = 0. \end{cases}$$

By the random payment rule, it holds that $\text{Prob}\{\Gamma_{i2}(\hat{\boldsymbol{x}}) \leq \hat{w}_i\} = 1$ for each i. Thus, we have constructed a well-defined random direct mechanism $\mathbf{\Gamma} = (\Gamma_{11}, \ldots, \Gamma_{n1}, \Gamma_{12}, \ldots, \Gamma_{n2})$. Obviously, the associated direct mechanism of $\mathbf{\Gamma}$ satisfies anonymity and monotonicity, and thus $\mathbf{\Gamma}$ is a standard random direct mechanism.

4. CONTINUUM-MASS HYBRID MECHANISM

Section 3 proposed a class of discrete hybrid mechanisms and defined their characteristics. However, it is usually difficult to compute these characteristics. Fortunately, the number of buyers and licenses are both large in practice and the discrete problem is close to the version with a continuum of mass. In addition, in a setting of continuum-mass buyers and items, it is relatively simple to compute the characteristics of a mechanism, because the mass of buyers with certain types and accordingly the auction price are usually determined. In this section, with the aim of helping the social planner to choose an optimal mechanism, we shall provide a continuum-mass mechanism for each hybrid mechanism and use its characteristics to approximate those of the hybrid mechanism.

4.1. Description of continuum-mass hybrid mechanisms

For each hybrid mechanism $H(r, m_1)$, we define its continuum-mass version hybrid mechanism as follows. A social planner wishes to assign a mass $\alpha = \frac{m}{n} \in (0,1)$ of vehicle licenses to a unit mass of buyers, in which $\alpha_1 = \frac{m_1}{n}$ mass and $\alpha - \alpha_1 = \frac{m-m_1}{n}$ mass of licenses are allocated by auction and lottery, respectively. All assumptions on buyers' values, budgets, and utility functions are similar as in Section 2. The continuum-mass hybrid mechanism is defined as follows.

Continuum-mass hybrid auction lottery mechanism for vehicle licenses

Step 1 **Announcement** The social planner announces the total license quota α, the auction quota $\alpha_1 \in [0, \alpha]$, the lottery quota $\alpha - \alpha_1$, and a reserve price r.

Step 2 **Registration** All buyers decide whether to register for the mechanism. At the end of registration, the social planner announces the mass of registered buyers β.

Step 3 **Auction** All registered buyers submit their bids, which should not be less than r.[13] A buyer wins the auction, obtains a license, and pays the equilibrium price p^e if her bid is no less than p^e.[14]

[13] If a registered buyer does not submit a bid, her bid is set as r by default.

[14] Let $D(p)$ be the mass of those buyers who bid above p. Then, the equilibrium price p^e is the price that satisfies $D(p^e) = \alpha_1$.

Step 4 **Lottery** Registered buyers who lose the auction enter the lottery in which $\alpha - \alpha_1$ mass of licenses is allocated. Each winner in lottery obtains a license and pays r.

Since the total license quota α is exogenously given, we shall use $H(r, \alpha_1)$ to denote such a continuum-mass hybrid mechanism and write its characteristics as $Ch(r, \alpha_1) = (Eq(r, \alpha_1), Ef(r, \alpha_1), Re(r, \alpha_1))$. We also use

$$\lambda = \begin{cases} \frac{\alpha - \alpha_1}{\beta - \alpha_1}, & \text{if } \beta > \alpha, \\ 1, & \text{if } \beta \leq \alpha, \end{cases}$$

to denote the probability of winning a license in the lottery.

In a continuum-mass hybrid mechanism $H(r, \alpha_1)$, we say a buyer with type $x = (v, w)$ *bids sincerely* whenever

1. she registers for the mechanism, if and only if, $\min\{v_i, w_i\} \geq r$;

2. when she is a registered buyer, she bids $\min\{\, v_i - \lambda(v_i - r),\, w_i \,\}$.

We have the following result on sincere bidding strategy as the counterpart of Theorem 1.

Theorem 2. *Each continuum-mass hybrid mechanism $H(r, \alpha_1)$ is ex-post individually rational. Moreover, in a continuum-mass hybrid mechanism $H(r, \alpha_1)$, for every buyer, it is a weakly dominant strategy to bid sincerely.*

From the result of Theorem 2, we assume in the following that every buyer bids sincerely in $H(r, \alpha_1)$. Then, only those buyers whose values and budgets are both no less than r would register for the hybrid mechanism, and thus the mass of registered buyers would be

$$\beta = \int_r^{\bar{w}} \int_r^{\bar{v}} \phi(v, w) dv dw.$$

Thus, there is a unique critical reserve price $r^* \in [0, \min\{\bar{v}, \bar{w}\}]$ such that

$$\alpha = \int_{r^*}^{\bar{w}} \int_{r^*}^{\bar{v}} \phi(v, w) dv dw,$$

because $\Phi(v, w)$ is assumed to be strictly increasing in v and w. Note that in the case that $\alpha_1 = \alpha$ and $r = 0$, r^* is also the auction price. Obviously, for a mechanism $H(r, \alpha_1)$ with $r < r^*$ it holds that $\beta > \alpha$, and for a mechanism $H(r, \alpha_1)$ with $r \geq r^*$ it satisfies $\beta \leq \alpha$. We further see that, for a mechanism $H(r, \alpha_1)$ with $r < r^*$ and a (possible) price $p \in [r, \min\{(1-\lambda)\bar{v} + \lambda r,\ \bar{w}\}]$, only those buyers with budgets no less than p and values no less than $\frac{p-\lambda r}{1-\lambda}$ would bid above p. Therefore, the mass of buyers bidding above p would be

$$D(p) = \int_p^{\bar{w}} \int_{v_p}^{\bar{v}} \phi(v, w) dv dw,$$

where $v_p = \frac{p-\lambda r}{1-\lambda}$. Thus, there also exists a unique equilibrium price $p^e \in [r, \min\{(1-\lambda)\bar{v} + \lambda r, \bar{w}\}]$ such that

$$\alpha_1 = \int_{p^e}^{\bar{w}} \int_{v^p}^{\bar{v}} \phi(v, w) dv dw,$$

where $v^p = \frac{1}{1-\lambda}(p^e - \lambda r)$. In summary, we obtain the following result.

Corollary 1. *If every buyer bids sincerely in $H(r, \alpha_1)$, then it holds that*

(1) The mass of registered buyers β satisfies $\beta = \int_r^{\bar{w}} \int_r^{\bar{v}} \phi(v, w) dv dw$.

(2) If $r < r^*$, an equilibrium price $p^e \in [r,\ \min\{(1-\lambda)\bar{v} + \lambda r,\ \bar{w}\}]$ exists that satisfies $\alpha_1 = \int_{p^e}^{\bar{w}} \int_{v^p}^{\bar{v}} \phi(v, w) dv dw$, where $v^p = \frac{1}{1-\lambda}(p^e - \lambda r)$ is the lowest value of all winning bidders in auction.

4.2. Characteristics of continuum-mass hybrid mechanisms

In this subsection, we shall present the formulas of the characteristics for continuum-mass hybrid mechanisms. We first consider the characteristics $Ch(r, \alpha_1)$ of a hybrid mechanism $H(r, \alpha_1)$ with $r < r^*$.

For such a hybrid mechanism, we have $\beta > \alpha$. Let $A = [r, \bar{v}] \times [r, p^e]$, $B = [r, v^p) \times [p^e, \bar{w}]$, $C = [v^p, \bar{v}] \times [p^e, \bar{w}]$ and $D = A \cup B$. We can show that, if all buyers bid sincerely, buyers in C win the auction,[15] and buyers in D enter the lottery. Figure 2 demonstrates such different winning patterns of buyers.

In the following, we consider efficiency, revenue, and equality of $H(r, \alpha_1)$ sequentially.

Efficiency $Ef(r, \alpha_1)$

[15] Here, "buyers in C" means that buyers whose types are in C, and the same below.

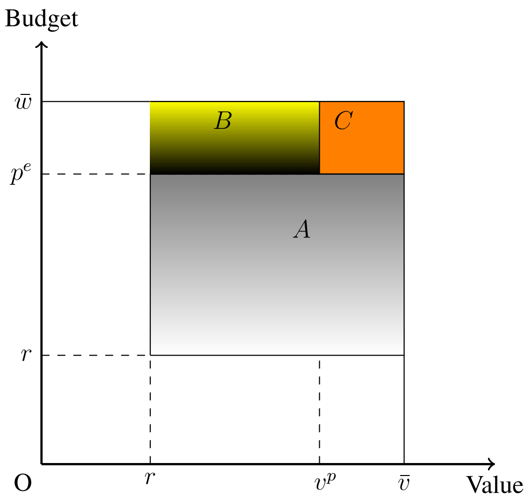


Figure 2: Winning patterns of buyers with different types

Let P_A, P_B, and P_C denote the masses of the buyers in A, B, and C, respectively, and let EV_A, EV_B, and EV_C represent the mean value of the buyers in A, B, and C, respectively. We then have that

$$P_A = \int_r^{p^e} \int_r^{\bar{v}} \phi(v,w)dvdw, \qquad EV_A = \frac{1}{P_A} \int_r^{p^e} \int_r^{\bar{v}} v\phi(v,w)dvdw,$$

$$P_B = \int_{p^e}^{\bar{w}} \int_r^{v^p} \phi(v,w)dvdw, \qquad EV_B = \frac{1}{P_B} \int_{p^e}^{\bar{w}} \int_r^{v^p} v\phi(v,w)dvdw,$$

and

$$P_C = \int_{p^e}^{\bar{w}} \int_{v^p}^{\bar{v}} \phi(v,w)dvdw, \qquad EV_C = \frac{1}{P_C} \int_{p^e}^{\bar{w}} \int_{v^p}^{\bar{v}} vd\,\Phi(v,w).$$

Thus, the mean value of winners in lottery EV_D can be written as

$$\begin{aligned} EV_D &= \frac{P_A}{P_A + P_B} \cdot EV_A + \frac{P_B}{P_A + P_B} \cdot EV_B \\ &= \frac{\lambda}{\alpha - \alpha_1} \cdot \Big(\int_r^{p^e} \int_r^{\bar{v}} vd\,\Phi(v,w) + \int_{p^e}^{\bar{w}} \int_r^{v^p} vd\,\Phi(v,w) \Big). \end{aligned}$$

Efficiency is measured by the mean value of all winning buyers. Therefore, it is the mean value of winners in auction (EV_C) weighted by the share of auction quota $\frac{\alpha_1}{\alpha}$ plus the mean value of lottery winners (EV_D) weighted by the lottery quota share $\frac{\alpha-\alpha_1}{\alpha}$. Formally, the efficiency measure can be written as

$$\begin{aligned} Ef(r,\alpha_1) &= \frac{\alpha_1}{\alpha}\cdot EV_C + \frac{\alpha-\alpha_1}{\alpha}\cdot EV_D \\ &= \frac{1}{\alpha}\cdot\int_{p^e}^{\bar{w}}\int_{v^p}^{\bar{v}} vd\,\Phi(v,w) \\ &\quad + \frac{\lambda}{\alpha}\cdot\Big(\int_r^{p^e}\int_r^{\bar{v}} vd\,\Phi(v,w) + \int_{p^e}^{\bar{w}}\int_r^{v^p} vd\,\Phi(v,w)\Big). \end{aligned} \tag{4.1}$$

Revenue $Re(r,\alpha_1)$

The revenue of the hybrid mechanism is induced from the price p^e paid by those winning buyers in the auction and the reserve price r paid by winners in the lottery. Therefore, the revenue measure, i.e., the mean payment for a license, can be written as

$$Re(r,\alpha_1) = \frac{\alpha_1}{\alpha}\cdot p^e + \frac{\alpha-\alpha_1}{\alpha}\cdot r. \tag{4.2}$$

Equality $Eq(r,\alpha_1)$

For a budget level $w\in[0,\bar{w}]$, recall that $p(w)$ represents the probability of winning a license for a buyer whose budget is w, and $P(w)$ represents the cumulative mass of licenses won by buyers with budgets no greater than w. First, it is clear that a buyer with budget $w\in[0,r)$ has no chance of winning a license. Second, for a buyer with $w\in[r,p^e)$, the probability that she is positioned in area A and enters the lottery is $1-F(r|w)$, and thus she can win a license with the probability $\lambda\big(1-F(r|w)\big)$. Third, for a buyer with budget $w\in[p^e,\bar{w}]$, the probability that she is positioned in area B and enters the lottery is $F(v^p|w)-F(r|w)$, whereas the probability that she is positioned in area C and wins the auction is $1-F(v^p|w)$, accordingly, her probability of winning a license is $\lambda\big(1-F(r|w)\big)+(1-\lambda)\big(1-F(v^p|w)\big)$. As a result, we can write $p(w)$ and $P(w)$ as

$$p(w) = \begin{cases} 0, & \text{if } w<r, \\ \lambda\big(1-F(r|w)\big), & \text{if } r\le w<p^e, \\ \lambda\big(1-F(r|w)\big)+(1-\lambda)\big(1-F(v^p|w)\big), & \text{if } p^e\le w\le\bar{w}, \end{cases}$$

and

$$P(w) = \begin{cases} 0, & \text{if } w < r, \\ \lambda \int_r^w \big(1 - F(r|t)\big) dG(t), & \text{if } r \le w < p^e, \\ \lambda \int_r^w \big(1 - F(r|t)\big) dG(t) & \\ +(1-\lambda) \int_{p^e}^w \big(1 - F(v^p|t)\big) dG(t), & \text{if } p^e \le w \le \bar{w}. \end{cases}$$

Consequently, according to equation (2.9), the equality measure can be expressed as

$$\begin{aligned} Eq(r, \alpha_1) &= 2\int_0^1 L(s)\, ds \; = \; \frac{2}{\alpha} \cdot \int_0^{\bar{w}} P(w) dG(w) \\ &= \frac{2\lambda}{\alpha} \cdot \int_r^{\bar{w}} \int_r^w \big(1 - F(r|t)\big) dG(t) dG(w) \\ &\quad + \frac{2(1-\lambda)}{\alpha} \cdot \int_{p^e}^{\bar{w}} \int_{p^e}^w \big(1 - F(v^p|t)\big) dG(t) dG(w). \end{aligned} \tag{4.3}$$

Finally, let us consider the characteristics $Ch(r, \alpha_1)$ of a hybrid mechanism $H(r, \alpha_1)$ with $r \ge r^*$. Such a mechanism satisfies $\beta \le \alpha$. This means that every buyer whose value and budget are both no less than r would register for the mechanism and win a license for sure. Therefore, the characteristics $Ch(r, \alpha_1)$ can be expressed as

$$\begin{aligned} Ef(r, \alpha_1) &= \frac{1}{\alpha} \cdot \int_r^{\bar{w}} \int_r^{\bar{v}} v d\Phi(v, w), \\ Re(r, \alpha_1) &= \frac{\beta}{\alpha} \cdot r, \qquad \text{and} \\ Eq(r, \alpha_1) &= \frac{2}{\beta} \cdot \int_r^{\bar{w}} \int_r^w \big(1 - F(r|t)\big) dG(t) dG(w). \end{aligned} \tag{4.4}$$

4.3. False-name bidding in continuum-mass hybrid mechanisms

In this subsection, we shall examine buyers' incentives to engage in false-name bidding in the continuum-mass hybrid mechanisms, and then provide a sufficient and necessary condition to prevent false-name bidding. Specifically, we shall analyze, for a given auction quota α_1, how the social planner chooses the proper reserve price r to curb false-name bidding. We first consider a simple case where a buyer bids under one false name.

Note that in any hybrid mechanism $H(r, \alpha_1)$, a buyer with budget $w < 2r$ has no incentive to bid under a false name because of the absolute budget constraint. Therefore, if $\bar{w} \neq +\infty$, the social planner can always set $r > \frac{1}{2}\bar{w}$ to prevent any buyer from bidding under a false name. However, this method cannot work if $\bar{w} = +\infty$. Moreover, if $r > \frac{1}{2}\bar{w} \geq r^*$, the characteristics $Ch(r, \alpha_1)$ will be impaired. Therefore, in the following, we shall consider some other methods of preventing false-name bidding, and thus assume that $r \leq \frac{1}{2}\bar{w}$.

Pick a hybrid mechanism $H(r, \alpha_1)$ with $r \leq \frac{1}{2}\bar{w}$. As in the previous subsection, we use C and D to denote the set of buyers who win the auction and the set of buyers who enter the lottery, when all buyers bid sincerely. Then, we see that when a buyer in C bids under her true identity, her expected payoff is $v - p^e$, and when a buyer in D bids under her true identity, her expected payoff is $\lambda(v-r)$. In addition, for a buyer with value $v \geq r$ and budget $w \geq 2r$, her expected payoff from bidding under a false name is $v - 2\lambda r - (1-\lambda)^2 v$.

When the hybrid mechanism $H(r, \alpha_1)$ satisfies $p^e > (1+\lambda)r$, we see that $\frac{r}{1-\lambda} < v^p < \bar{v}$ and $r < \frac{p^e - 2\lambda r}{(1-\lambda)^2}$, thus the set

$$E = D \bigcap \left(\frac{r}{1-\lambda}, \bar{v}\right] \times [2r, \bar{w}]$$

and the set

$$F = C \bigcap \left[r, \frac{p^e - 2\lambda r}{(1-\lambda)^2}\right) \times [2r, \bar{w}]$$

are both well-defined. We can further show that: (i) the sets E and F are both non-empty; (ii) for every buyer in E, her net surplus from bidding under a false name is

$$[v - 2\lambda r - (1-\lambda)^2 v] - \lambda(v - r) \;=\; \lambda(1-\lambda)v - \lambda r \;>\; 0;$$

and (iii) for every buyer in F, her net surplus from bidding under a false name is

$$[v - 2\lambda r - (1-\lambda)^2 r] - (v - p^e) \;=\; p^e - 2\lambda r - (1-\lambda)^2 v \;>\; 0.$$

Therefore, all buyers in $E \cup F$ have incentives to bid under a false name. Specifically, those buyers in E prefer bidding under two identities compared to their original strategy of participating in the lottery with one identity, and those buyers in F prefer bidding under two identities compared to their original

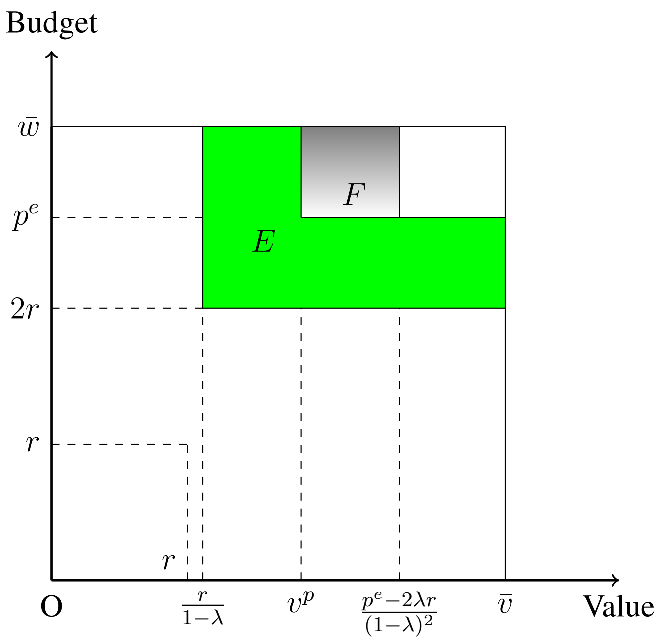


Figure 3: Buyers who have incentives to engage in false-name bidding in a hybrid mechanism $H(r, \alpha_1)$ with $p^e > 2r > (1+\lambda)r$

strategy of winning in the auction. Figure 3 demonstrates such different patterns of buyers who have incentives to bid under a false name in a hybrid mechanism with $p^e > 2r > (1+\lambda)r$.

As argued above, the social planner must set r such that $p^e \leq (1+\lambda)r$ to prevent buyers from bidding under a false name. Indeed, there always exists some $r \in (0, r^*)$ such that $p^e \leq (1+\lambda)r$ because $p^e < (1+\lambda)r$ for $r \to r^{*-}$. In fact, we can further show that $p^e \leq (1+\lambda)r$ is also a sufficient condition to prevent buyers from bidding under any number of false names. Formally, we have the following result.

Theorem 3. *In a continuum-mass hybrid mechanism $H(r, \alpha_1)$ with $2r \leq \bar{w}$, no buyer has an incentive to bid under any number of false names if and only if $p^e \leq (1+\lambda)r$.*

The above analysis is based on the assumption that a buyer's cost for holding an extra license is just the reserve price r. In practice, besides adjusting

r, the social planner can implement other policies to increase buyers' cost of holding extra licenses, and further curb false-name bidding. For instance, Beijing has adopted resale prohibitions and identity regulations, in addition to a stipulation that winners have to buy vehicles within a limited time period after the lottery.

5. PROBABILITY ALLOCATION MECHANISM

We have provided a class of hybrid mechanisms that are ex-post individually rational. In this section, we shall relax the requirement of ex-post individual rationality, and explore an allocation mechanism that can improve upon the hybrid mechanisms in all three factors of efficiency, equality, and revenue. In the following, we shall refer to it as *the probability allocation mechanism*. This mechanism's associated interim allocation mechanism is essentially similar to the allocation rule of Ausubel (2004),[16] and its random payment rule is similar to randomized extraction by Bhattacharya et al. (2010). Before defining this mechanism, we first introduce some notations.

Let $\hat{\boldsymbol{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \in \mathcal{X}$, where $\hat{x}_i = (\hat{v}_i, \hat{w}_i)$ for each $i \in \mathcal{N}$, be a report profile of buyers. Then every buyer i can be viewed as holding a left-continuous demand function

$$d_i(p) = \begin{cases} 1, & \text{if } p \leq \hat{v}_i \text{ and } p \leq \hat{w}_i, \\ \frac{\hat{w}_i}{p}, & \text{if } p \leq \hat{v}_i \text{ and } p > \hat{w}_i, \\ 0, & \text{if } p > \hat{v}_i. \end{cases} \tag{5.1}$$

Thus, there is a left-continuous total demand function $D(p) = \sum_{i=1}^{n} d_i(p)$, and every buyer i faces a left-continuous **residual supply function** $s^{-i}(p) = \max\left\{m - \sum_{j \neq i} d_j(p), 0\right\}$, where m is the license quota. Since $D(p)$ is a nonincreasing function with $D(0) = n > m$ and $D(\bar{v}) = 0 < m$, there always exists a critical price $p^* = p(\hat{\boldsymbol{x}}) \in [0, \bar{v}]$ such that $p^* = \max\{p : D(p) \geq m\}$. Let $D(p^{*+}) = \lim_{p \to p^{*+}} D(p)$ be the right-sided limit of $D(p)$ with p approaching p^* from the right.

[16] Dobzinski et al. (2012) and Bhattacharya et al. (2010) propose an adaptive clinching auction allocating the probability of winning objects based on Ausubel (2004). Their mechanism exhausts most winning buyers' budgets but is far more complex than our probability allocation mechanism.

Having prepared these notation, we define the probability allocation mechanism as follows. For each report profile $\hat{\boldsymbol{x}}$ and each buyer $i \in \mathcal{N}$, the interim winning probability $Q_i(\hat{\boldsymbol{x}})$ and the expected payment $M_i(\hat{\boldsymbol{x}})$ of buyer i are given by

$$Q_i(\hat{\boldsymbol{x}}) = \begin{cases} \frac{m - D(p^{*+})}{D(p^*) - D(p^{*+})} \cdot d_i(p^*), & \text{if } v_i = p^*, \\ d_i(p^*), & \text{if } v_i \neq p^*, \end{cases}$$

and[17]

$$M_i(\hat{\boldsymbol{x}}) = \int_0^{p^*} \big(Q_i(\hat{\boldsymbol{x}}) - s^{-i}(p)\big)\, dp.$$

According to the Birkhoff-von Neumann theorem (Budish et al., 2013), there is a random assignment $\boldsymbol{\Gamma}_1(\hat{\boldsymbol{x}}) = (\Gamma_{11}(\hat{\boldsymbol{x}}), \Gamma_{21}(\hat{\boldsymbol{x}}), \ldots, \Gamma_{n1}(\hat{\boldsymbol{x}})) \in \mathcal{F}(\Pi)$ such that $E[\Gamma_{i1}(\hat{\boldsymbol{x}})] = Q_i(\hat{\boldsymbol{x}})$ for all $i \in \mathcal{N}$.

Note that for some buyer i, it may hold that $0 < Q_i(\hat{\boldsymbol{x}}) < 1$ and $M_i(\hat{\boldsymbol{x}}) < \hat{w}_i$. Therefore, if we take buyer i's expected payment $M_i(\hat{\boldsymbol{x}})$ as a deterministic payment, then she has an incentive to over-report her budget. To prevent such over-reportings, we, similar to Bhattacharya et al. (2010), need a random payment rule with a positive probability that buyer i will be required to pay the full amount of her reported budget. We construct a random payment for each buyer i satisfying $E[\Gamma_{i2}(\hat{\boldsymbol{x}})] = M_i(\hat{\boldsymbol{x}})$, Prob$\{\Gamma_{i2}(\hat{\boldsymbol{x}}) = \hat{w}_i\} > 0$ and Prob$\{\Gamma_{i2}(\hat{\boldsymbol{x}}) \leq \hat{w}_i\} = 1$ as

$$\Gamma_{i2}(\hat{\boldsymbol{x}}) = \begin{cases} \hat{w}_i, & \text{with probability } \frac{M_i(\hat{\boldsymbol{x}})}{\hat{w}_i}, \\ 0, & \text{with probability } 1 - \frac{M_i(\hat{\boldsymbol{x}})}{\hat{w}_i}. \end{cases}$$

We thus obtain a well-defined random direct mechanism:

$$\boldsymbol{\Gamma} = (\Gamma_{11}, \ldots, \Gamma_{n1}, \Gamma_{12}, \ldots, \Gamma_{n2}),$$

which is referred to as the probability allocation mechanism. Note that since its associated direct mechanism (Q, M) satisfies anonymity and monotonicity, $\boldsymbol{\Gamma}$ is a standard random direct mechanism. In this mechanism, each buyer i's random payment $\Gamma_{i2}(\hat{\boldsymbol{x}})$ is independent of her random assignment $\Gamma_{i1}(\hat{\boldsymbol{x}})$, and

[17] We can use an example to illustrate the allocation rule. Suppose $m = 10000$, $D(p^*) = 10000.25$ and $D(p^{*+}) = 9999.5$, and suppose that buyer i is the "jumping" buyer with value p^* and budget $0.75p^*$. Then, buyer i is assigned a license with the probability $\frac{10000-9999.5}{10000.25-9999.5} \cdot \frac{0.75p^*}{p^*} = 0.5$.

thus $\boldsymbol{\Gamma}$ is not ex-post individually rational. Unlike in our hybrid mechanisms, for some report profile $\hat{\boldsymbol{x}}$ and some buyer i, her payment probability $\frac{M_i(\hat{\boldsymbol{x}})}{\hat{w}_i}$ may be higher than her interim winning probability $Q_i(\hat{\boldsymbol{x}})$ in $\boldsymbol{\Gamma}$. Thus, we cannot construct a random allocation rule such that only those buyers who eventually obtain licenses are required to pay. Thus, $\boldsymbol{\Gamma}$ cannot be modified to be ex-post individually rational. Fortunately, $\boldsymbol{\Gamma}$ satisfies some other desirable properties as the following theorem states.

Theorem 4. *The probability allocation mechanism* $\boldsymbol{\Gamma}$ *is interim individually rational and weakly dominant strategy incentive compatible.*

In the probability allocation mechanism, indivisible licenses are treated as divisible goods. Namely, this mechanism sells license winning probabilities to buyers. Therefore, the probability allocation mechanism further relaxes buyers' budget constraints, and thus it can achieve better characteristics than our hybrid mechanisms. In fact, the numerical analysis in the next section confirms these better characteristics. Moreover, it is worth mentioning that the probability allocation mechanism is not ex-post individually rational, thus it is not covered in Che et al. (2013)'s discussion of the efficiency-optimal mechanism.

6. NUMERICAL ANALYSIS OF THE CHARACTERISTICS

Section 4 defined the continuum-mass mechanism $H(r, \alpha_1)$ for every hybrid mechanism $H(r, m_1)$, and presented the formulas for its characteristics $Ch(r, \alpha_1)$. In this section, we shall first use simulation and numerical computation to test if $Ch(r, \alpha_1)$ is a good approximation of $Ch(r, m_1)$. Second, to enable the social planner to choose, we shall plot the attainable characteristics of the hybrid mechanisms through numerical computation. Third, using these figures, we shall compare the attainable characteristics of the hybrid mechanisms with those of the existing license allocation mechanisms in China, and with those of the probability allocation mechanism.

Environment of numerical analysis

In the following, we set the number of potential buyers as $n = 100000$ and the license quota as $m = 10000$, and thus the mass of license quota $\alpha = \frac{1}{10}$. For simplicity, the value and budget of each buyer is set to be mutually independently and identically distributed. We indeed conduct the numerical analysis under two distributions: the uniform distribution on $[0, 10000]$ and the exponential distribution with a mean value of 5000, respectively.

Since Theorem 1 indicates that all buyers have incentives to bid sincerely, we can use the simulation method to examine the characteristics of discrete hybrid mechanisms. Specifically, for a hybrid mechanism $H(r, m_1)$, we first draw 1000 profiles $\{\boldsymbol{x}^{(k)} \,|\, k = 1, \cdots, 1000\}$ according to the uniform distribution (or, the exponential distribution). For each profile $\boldsymbol{x}^{(k)}$, according to Theorem 1 and the rule of $H(r, m_1)$ we can obtain an interim allocation $(\mathrm{Q}(\boldsymbol{x}^{(k)}), \mathrm{M}(\boldsymbol{x}^{(k)}))$. Accordingly, we can further obtain the interim efficiency $Ef(\boldsymbol{x}^{(k)})$, the interim revenue $Re(\boldsymbol{x}^{(k)})$, and the vector $\hat{\boldsymbol{Q}}(\boldsymbol{x}^{(k)})$ of interim winning probability for all buyers ranked by their budgets from low to high. We then use $\frac{1}{1000}\sum_{k=1}^{1000} Ef(\boldsymbol{x}^{(k)})$ to represent efficiency $Ef(r, m_1)$, and use $\frac{1}{1000}\sum_{k=1}^{1000} Re(\boldsymbol{x}^{(k)})$ to represent revenue $Re(r, m_1)$. By formula (1.6), we calculate the equality measure $Eq(r, m_1)$ from

$$\overline{\boldsymbol{Q}} = \frac{1}{1000} \cdot \sum_{k=1}^{1000} \hat{\boldsymbol{Q}}(\boldsymbol{x}^{(k)}).$$

To test whether it is robust to approximate $Ch(r, m_1)$ with $Ch(r, \frac{m_1}{n})$, we uniformly draw 121 parameters from

$$P_1 = \Big\{(r, m_1) = 1000(s, t) \mid s, t = 0, 1, \ldots, 10\Big\}$$

and compare each $Ch(r, m_1)$ with its continuum-mass counterpart $Ch(r, \frac{m_1}{n})$. The characteristics $Ch(r, m_1)$ are computed through the simulation method described above, and the characteristics $Ch(r, \frac{m_1}{n})$ are computed according to the formulas in Subsection 4.2. The relative difference between efficiency $Ef(r, m_1)$ and $Ef(r, \frac{m_1}{n})$ is calculated as

$$\frac{Ef(r, \frac{m_1}{n}) - Ef(r, m_1)}{Ef(r, m_1)},$$ [18]

and the relative differences in equality and revenue are similarly computed. We find that the absolute values of the relative differences between these characteristics $Ch(r, m_1)$ and $Ch(r, \frac{m_1}{n})$ with economically meaningful $r <$

[18] In the trival case of $r \geq \min\{\bar{v}, \bar{w}\}$, it holds that $Ef(r, m_1) = 0$. In this case, we set the relative difference between $Ef(r, m_1)$ and $Ef(r, \frac{m_1}{n})$ as 0.

r^* are usually less than 0.01%, while the largest absolute values of relative differences are those between $Eq(r, m_1)$ and $Eq(r, \frac{m_1}{n})$ with $r \geq r^*$, and the scale of these largest absolute values is approximately 0.1%. The comparison results are listed in Appendix B.

To plot the attainable characteristics of hybrid mechanisms, we uniformly draw 10201 parameters from

$$P_2 = \Big\{(r, m_1) = 100(s,t) \mid s,t = 0,1,\ldots,100\Big\}.$$

Since we have shown that the characteristics of continuum-mass mechanisms can approximate the characteristics of discrete mechanism well, we can substitute $Ch(r, m)$ with $Ch(r, \frac{m_1}{n})$. The characteristics of the hybrid mechanisms with parameters selected from P_2 are computed numerically and are plotted in blue in Figures 4, 5, and 6. In addition, the characteristics achievable by those hybrid mechanisms that are false-name-bidding-proof, i.e., satisfy $p^e \leq (1+\lambda)r$, are plotted in yellow in those figures.

Recall that the current Shanghai mechanism $M(c)$ with a warning price c can be induced from our hybrid mechanisms. Formally, we have

$$M(c) = \begin{cases} H(c, 0), & \text{if } c \leq p^e, \\ H(0, \frac{m}{n}), & \text{if } c > p^e. \end{cases}$$

Let $Ch(c)$ denote the characteristics of $M(c)$. Then we have

$$Ch(c) = \begin{cases} Ch(c, 0), & \text{if } c \leq p^e, \\ Ch(0, \frac{m}{n}), & \text{if } c > p^e. \end{cases}$$

We uniformly draw the parameter c from

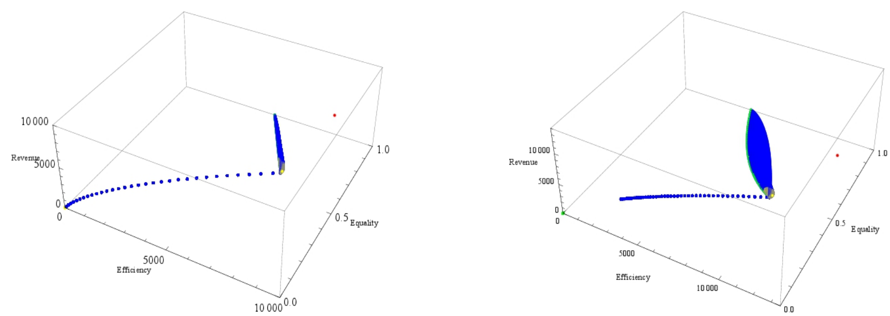
$$P_3 = \Big\{100s \mid s = 0,1,\ldots,100\Big\},$$

and plot the attainable characteristics of the current Shanghai mechanisms $Ch(c)$ in green in Figures 4, 5, and 6.

According to Theorem 4, the probability allocation mechanism is an IC mechanism, and thus we can compute its characteristics through simulation.[19]

[19] Since every buyer's winning probability is nondecreasing in her reported budget and value in the probability allocation mechanism, we see that the equality measure of the probability allocation mechanism is well-defined according to Theorem 4 and Assumption 1.

The computed characteristics are plotted in red in Figures 4, 5, and 6. Figure 4 presents the 3-D characteristics of different mechanisms. Since these 3-D graphs are difficult to comprehend intuitively, we plot the 2-D efficiency-equality characteristics of different mechanisms with uniformly distributed buyers' types in Figure 5, and with exponentially distributed buyers' types in Figure 6. Since wealth of buyers are usually supposed to be subject to Pareto distribution in literature, and that the Pareto distributions are closer to exponential distributions than uniform distributions, buyers' values and budgets are more likely to be exponentially distributed in practice. Therefore, in the following, we shall focus our discussion on Figure 6.



(a) 3-D characteristics of different mechanisms with uniform distribution

(b) 3-D characteristics of different mechanisms with exponential distribution

Figure 4: 3-D characteristics of different mechanisms

In Figure 6, we highlight the 2-D characteristics of several special mechanisms: (i) b — the Beijing mechanism, i.e., a pure lottery with no reserve price $H(0,0)$; (ii) s — the Shanghai auction before July 2013, i.e., a pure auction with no reserve price $H(0,0.1)$; (iii) e — the hybrid mechanism with the highest efficiency;[20] (iv) p — the probability allocation mechanism. These special mechanisms and their characteristics are listed in Table 1.

Figure 6 illustrates that each current Shanghai mechanism $M(c)$ generates the lowest efficiency $Ef(c,0)$ in all hybrid mechanisms that keep the same equality level $Eq(c,0)$. For instance, the current Shanghai mechanism $M(2500) = H(2500,0)$ yields the 2-D characteristics $d = (7500, 0.607)$.

[20] Here, the "highest" and "lowest" are among all hybrid mechanisms with parameters in P_2.

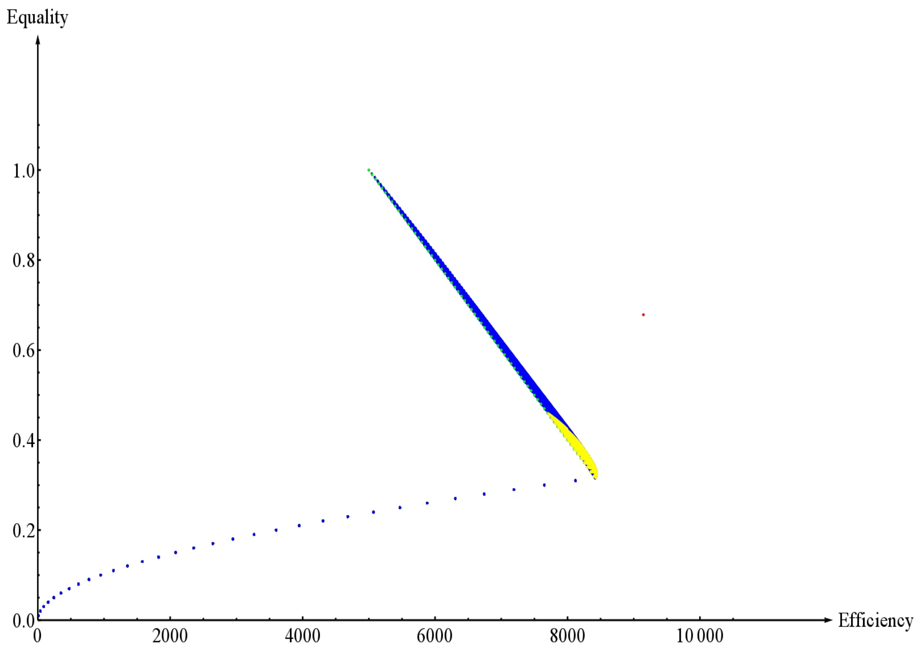


Figure 5: 2-D characteristics of different mechanisms with uniform distribution

However, compared with $M(2500)$, the hybrid mechanism $H(0,\ 0.055)$ brings the 2-D characteristics of $d' = (8890.8,\ 0.603)$, raising efficiency by 18.5%. The policy implication of our finding is profound: regardless of the social planner's objective, the current Shanghai mechanism, as a price ceiling mechanism, is not a wise choice.

Although the pure auction is usually viewed as the most efficient mechanism, Figure 6 demonstrates that it is less efficient than many other hybrid mechanisms. In theory, a hybrid mechanism provides an opportunity of lottery for buyers, and thus buyers will discount their bids in auction. Thus, buyers' budget constraints are relaxed as in the first-price auction with budget constraints; see e.g., Che & Gale (1998). Therefore, a hybrid mechanism may achieve higher efficiency than the pure auction. In fact, Che et al. (2013) have highlighted that in the presence of budget constraints, the efficiency-optimal mechanism always entails some random assignment. Our numerical analysis confirms this result. Specifically, among all hybrid mechanisms with parameters selected from P_2, the hybrid mechanism $H(5400,\ 0.076)$ achieves the highest efficiency. Comparing the characteristics of the pure auction $H(0, 0.1)$

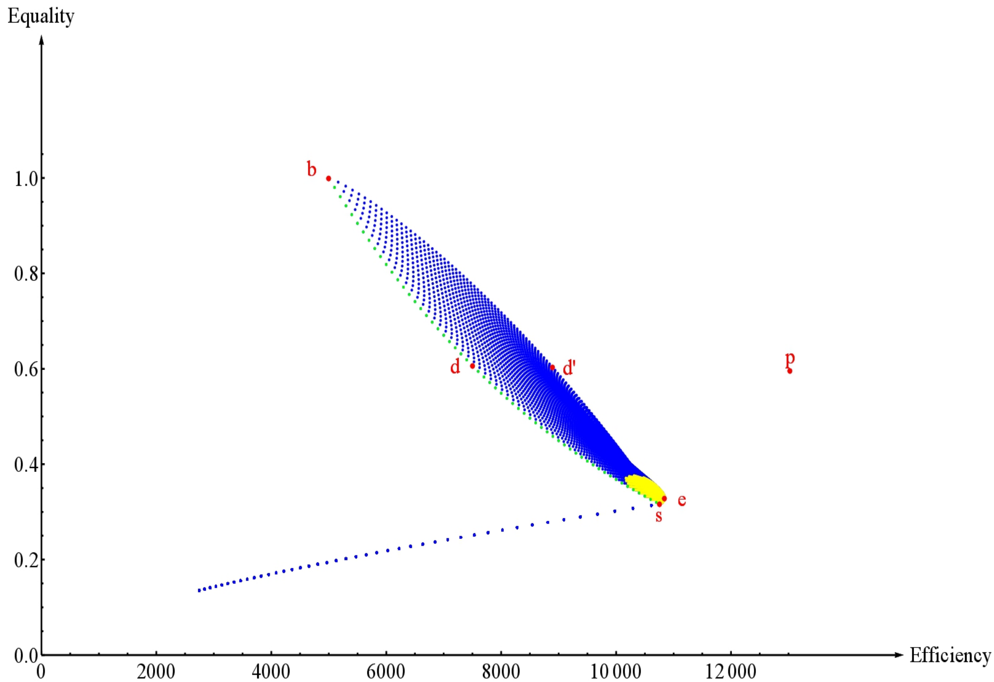


Figure 6: 2-D characteristics of different mechanisms with exponential distribution

Table 1: Characteristics of several special mechanisms with exponential distribution

$H(r, \alpha_1)$	(Ef, Eq)	Re	β	p^e	v^p	λ
$H(0,\ 0.000)$	$b =$ (5000.0, 1.000)	0.0	1.0000	-	-	0.1000
$H(0,\ 0.100)$	$s =$ (10756.5, 0.316)	5756.5	1.0000	5756.5	5756.5	0.3162
$H(5400,\ 0.076)$	$e =$ (10844.4, 0.328)	5844.4	0.1153	5984.7	6900.4	0.6103
PA mechanism	$p =$ (13024.7, 0.616)	8027.0	-	-	-	-
$H(2500,\ 0.000)$	$d =$ (7500.0, 0.607)	2500.0	0.3679	-	-	0.2718
$H(0,\ 0.055)$	$d' =$ (8890.8, 0.603)	3890.8	1.0000	7074.2	7427.9	0.0476

with those of $H(5400,\ 0.076)$ provides a hint of how a hybrid mechanism yields higher efficiency than the pure auction.

Figure 7 demonstrates the different types of winning buyers in the pure auction mechanism $H(0, 0.1)$ and the hybrid mechanism $H(5400,\ 0.076)$. In $H(0, 0.1)$, the auction price is $r^* = 5756$, and buyers in $A \cup B$ win the auction. In $H(5400,\ 0.076)$, the reserve price is $r = 5400$, the equilibrium price is $p^e = 5985$, and the critical value level below which no buyer wins the auction is $v^p = 6900$. In this mechanism, buyers in A still win the auction, and buyers in $B \cup C$ participate in the lottery. Figure 7 illustrates that B mainly comprises buyers with relatively low values and high budgets, which results in a lower mean value for buyers in B than for buyers in C. Indeed, the mean value for buyers in C 7967.3 is much higher than the mean value for buyers in B 7134.2. As a result, the efficiency of $H(5400,\ 0.076)$ is higher than the efficiency of the pure auction $H(0, 0.1)$.

The above Figures 4, 5, and 6 also show that the probability allocation mechanism achieves much higher efficiency and revenue than the hybrid mechanisms. In fact, from Table 1 we see that the probability allocation mechanism increases efficiency by 20.11% and raises revenue by 37.35% compared with the most efficient hybrid mechanism $H(5400,\ 0.076)$. Figure 6 further shows that point p — the 2-D characteristics of the probability allocation mechanism, lies far beyond the attainable set of hybrid mechanisms' characteristics. Therefore, although the probability allocation mechanism is not ex-post individually rational and can hardly be put into practice, it may provide inspiration for finding other mechanisms that are ex-post individually rational and weakly dominant strategy incentive compatible, and yield better characteristics than our hybrid mechanisms.

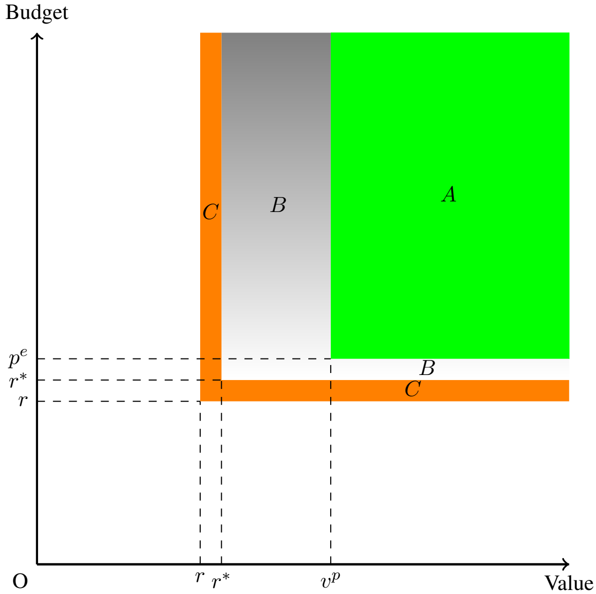


Figure 7: Buyers who can win licenses in $H(0, 0.1)$ and $H(5400, 0.076)$

7. CONCLUSION AND POLICY IMPLICATIONS

In this study, we examine the equality of public resource allocation mechanisms and introduce an equality measure as a new evaluation criterion for them in a multi-unit auction model with budget constraints. Our equality measure describes the difference in object obtaining opportunities among buyers with different wealth levels. As an application, we study vehicle license allocations in China. We especially propose a class of hybrid auction-lottery mechanisms to evaluate and improve upon China's vehicle license allocation mechanisms from the criteria of efficiency, equality and revenue, in a unified framework. For helping the social planner to evaluate hybrid mechanisms, we also provide a relative continuum-mass hybrid mechanism for each hybrid mechanism. In addition, we provide a probability allocation mechanism as a benchmark to compare with our hybrid mechanisms. Finally, using numerical analysis, we evaluate the characteristics of the hybrid mechanisms, compare different license allocation mechanisms in China, and provide useful insights into the improvement of vehicle license allocations in China. Our study can be widely applied to allocations of different public resources.

Several issues need to be addressed in future studies. First, it is both important and interesting to explore mechanisms that are no longer detail free, while achieve better characteristics than our hybrid mechanisms, and maintain simplicity of implementation, ex-post individual rationality, and incentive compatibility. Second, the possibility of extending our equality measure to the study of set-aside auctions and affirmative action in school choice can be discussed. Third, we just provide one equality measure in this paper. It is also an interesting question whether there are more reasonable equality measures based on the information about object obtaining opportunities of buyers with different budgets.

Appendix A: Proofs

A.1. Proof of Lemma 1

By the monotonicity of a standard direct mechanism and equation (1.1), we see that $q(v,w)$ is nondecreasing in both v and w. Therefore, by Assumption 1, for any $w,\ w' \in [0,\bar{w}]$ such that $w \le w'$, we have

$$\begin{aligned} p(w) = \int_0^{\bar{v}} q(v,w)dF(v|w) &\le \int_0^{\bar{v}} q(v,w)dF(v|w') \\ &\le \int_0^{\bar{v}} q(v,w')dF(v|w') = p(w'). \end{aligned}$$

Thus, $p(w)$ is a nondecreasing function. Note that for any $j \le j'$,

$$\begin{aligned} G_{(j)}(w) &= \sum_{i=j}^{n} \binom{n}{i} G^i(w)[1-G(w)]^{n-i} \\ &\ge \sum_{i=j'}^{n} \binom{n}{i} G^i(w)[1-G(w)]^{n-i} = G_{(j')}(w). \end{aligned}$$

Therefore, we further have

$$\overline{Q}_{(j)} = \int_0^{\bar{w}} p(w)dG_{(j)}(w) \le \int_0^{\bar{w}} p(w)dG_{(j')}(w) = \overline{Q}_{(j')}.$$

□

A.2. Proof of Theorem 1

We only prove the second part of the theorem. Suppose buyer i's type is $x_i = (v_i, w_i)$. According to the rule of the hybrid mechanisms, each buyer's license payment is always no less than r. Therefore, when $\min\{v_i, w_i\} < r$, a rational buyer i will not register for the mechanism no matter how other buyers bid. In the following, we assume $\min\{v_i, w_i\} \geq r$. Note that if buyer i registers for the hybrid mechanism, she can at least get an expected payoff $\lambda(v_i - r) \geq 0$ by bidding r. Therefore, we just consider that buyer i registers for the mechanism. Let $b_{-i}^{(m_1)}$ denote the m_1-th highest bid of all other buyers (if $n_1 < m_1$, set $b_{-i}^{(m_1)} = r$), and let $b_i(v_i, w_i) = \min\{\, v_i - \lambda(v_i - r),\, w_i \,\}$ be buyer i's sincere bid. In the following, we shall show that buyer i cannot improve her payoff by making a bid $b \geq r$ other than $b_i(v_i, w_i)$ in two cases.

Case 1, $b_i(v_i, w_i) \geq b_{-i}^{(m_1)}$. By submitting her sincere bid, buyer i wins the auction and receives a payoff $v_i - b_{-i}^{(m_1)} \geq \lambda(v_i - r)$. If she submits a bid $b \geq b_{-i}^{(m_1)}$, she still wins the auction and obtains the same payoff. If she bids $b < b_{-i}^{(m_1)}$, she enters the lottery and obtains a payoff $\lambda(v_i - r) \leq v_i - b_{-i}^{(m_1)}$. Therefore, buyer i receives the highest payoff by bidding $b_i(v_i, w_i)$ in this case.

Case 2, $b_i(v_i, w_i) < b_{-i}^{(m_1)}$. In this case, it satisfies that $w_i < b_{-i}^{(m_1)}$ or $v_i - b_{-i}^{(m_1)} < \lambda(v_i - r)$. By bidding sincerely, buyer i enters the lottery and obtains an expected payoff $\lambda(v_i - r)$. If buyer i bids $b \geq b_{-i}^{(m_1)}$, she wins the auction and obtains a payoff no greater than than $\lambda(v_i - r)$. If she makes a bid $b < b_{-i}^{(m_1)}$, she still enters the lottery and receives the same payoff. Therefore, buyer i cannot increase her expected payoff by any other bid.

To sum up, no matter how other buyers bid, buyer i obtains the highest expected payoff if she bids sincerely. □

A.3. Proof of Theorem 3

According to the argument in Subsection 4.3, when $p^e > (1 + \lambda)r$, the sets E and F are both non-empty, and all buyers in E and F have incentives to bid under a false name. Therefore, $p^e \leq (1 + \lambda)r$ is a necessary condition to prevent false-name bidding for a hybrid mechanism $H(r, \alpha_1)$ with $2r \leq \bar{w}$. In the following, we shall show that it is also a sufficient condition for preventing buyers from bidding under any number of false names.

Assume $p^e \leq (1+\lambda)r$. Note that only a buyer with value $v \geq r$ and budget $w \geq kr$ may have an incentive to bid under $k - 1$ ($k \in \mathbb{Z}_+$ and $k \geq 2$) false names, and such a buyer's expected payoff from bidding under $k - 1$ false names is $[1 - (1 - \lambda)^k]v - k\lambda r$. Thus, for such a buyer in D, her net surplus from bidding under $k-1$ false names is

$$\begin{aligned}
&[1-(1-\lambda)^k]v - k\lambda r - \lambda(v-r)\\
&= (1-\lambda)[1-(1-\lambda)^{k-1}]v - (k-1)\lambda r\\
&\le (1-\lambda)[(k-1)\lambda]v - (k-1)\lambda r\\
&\le (k-1)\lambda[(1-\lambda)\cdot\frac{p^e-\lambda r}{1-\lambda} - r]\\
&= (k-1)\lambda[p^e-(1+\lambda)r]\\
&\le 0.
\end{aligned}$$

For such a buyer in C, her net surplus from bidding under $k-1$ false names is

$$\begin{aligned}
&[1-(1-\lambda)^k]v - k\lambda r - (v-p^e)\\
&= p^e - (1-\lambda)^k v - k\lambda r\\
&\le p^e - (1-\lambda)^k \cdot \frac{p^e-\lambda r}{1-\lambda} - k\lambda r\\
&= [1-(1-\lambda)^{k-1}](p^e-\lambda r) - (k-1)\lambda r\\
&\le [(k-1)\lambda](p^e-\lambda r) - (k-1)\lambda r\\
&= (k-1)\lambda[p^e-(1+\lambda)r]\\
&\le 0.
\end{aligned}$$

We therefore see that all buyers in $C \cup D$ have no incentives to bid under any number of false names. In addition, it is clear that every buyer not in $C \cup D$ will not register for the mechanism, and has no incentive to bid under false names. Consequently, we have proved that $p^e \le (1+\lambda)r$ is a sufficient condition for preventing buyers from bidding under any number of false names. □

A.4. Proof of Theorem 4

By the probability allocation mechanism rule, no matter what other buyers report, each buyer's unit expected payment for her assigned license never exceeds her reported value, and her ex-post payment never exceeds her reported budget. Therefore, the probability allocation mechanism is interim individually rational. We then proceed to prove that the probability allocation mechanism is weakly dominant strategy incentive compatible.

Suppose the report profile of other buyers is $\hat{\boldsymbol{x}}_{-i}$. Then, buyer i faces a fixed nondecreasing residual supply function $s^{-i}(p)$. To simplify the notation, let $p^* = p(x_i, \hat{\boldsymbol{x}}_{-i})$, $q_i^* = Q_i(x_i, \hat{\boldsymbol{x}}_{-i})$ and $u_i^* = u_i(Q_i(x_i, \hat{\boldsymbol{x}}_{-i}), M_i(x_i, \hat{\boldsymbol{x}}_{-i}), x_i)$

denote the equilibrium price, the winning probability and expected utility of buyer i when she reports her true type x_i, respectively. And use $p' = p(x'_i, \hat{\boldsymbol{x}}_{-i})$, $q'_i = Q_i(x'_i, \hat{\boldsymbol{x}}_{-i})$ and $u'_i = u_i(Q_i(x'_i, \hat{\boldsymbol{x}}_{-i}), M_i(x'_i, \hat{\boldsymbol{x}}_{-i}), x_i)$ to denote the equilibrium price, the winning probability and expected utility of buyer i when she strategically reports her type $x'_i = (v'_i, w'_i)$, respectively. Since $\text{Prob}\{\Gamma_{i2}(x'_i, \hat{\boldsymbol{x}}_{-i}) = \hat{w}_i\} > 0$, buyer i cannot get a greater expected utility $u'_i > u^*_i$ by reporting a higher budget $w'_i > w_i$. To prove this theorem, it is sufficient to show $u'_i \le u^*_i$ for all $x'_i = (v'_i, w'_i)$ with $w'_i \le w_i$.

From the expected payment rule $M_i(\cdot)$, the expected utilities u^*_i and u'_i can be written as

$$\begin{aligned} u^*_i &= q^*_i v_i - M_i(x_i, \hat{\boldsymbol{x}}_{-i}) = q^*_i(v_i - p^*) + \int_0^{p^*} s^{-i}(p)\,dp, \\ u'_i &= q'_i v_i - M_i(x'_i, \hat{\boldsymbol{x}}_{-i}) = q'_i(v_i - p') + \int_0^{p'} s^{-i}(p)\,dp. \end{aligned}$$

By the interim assignment rule $Q_i(\cdot)$, we can show that $s^{-i}(\bar{p}) \le q'_i \le s^{-i}(\hat{p})$ for all prices $\bar{p}$ and $\hat{p}$ such that $\bar{p} \le p' < \hat{p}$. Therefore, it satisfies that

$$\begin{aligned} u'_i - u^*_i &= (q'_i - q^*_i)(v_i - p^*) - \int_{p^*}^{p'} [q'_i - s^{-i}(p)]\,dp \\ &\le (q'_i - q^*_i)(v_i - p^*). \end{aligned}$$

Note that in the case of $v_i < p^*$, it holds that $q^*_i = 0$, and hence $u'_i - u^*_i \le q'_i(v_i - p^*) \le 0$. Next, in the case of $p^* < v_i$, it holds $q^*_i = \min\{1, \frac{w_i}{p^*}\}$. Thus, for any $x'_i = (v'_i, w'_i)$ with $w'_i \le w_i$, if $p' \ge p^*$ then $q'_i \le \min\{1, \frac{w'_i}{p'}\} \le \min\{1, \frac{w_i}{p^*}\} = q^*_i$, if $p' < p^*$ then $q'_i < s^{-i}(p^*) \le q^*_i$, and so it always holds $q'_i \le q^*_i$. Hence, $u'_i - u^*_i \le (q'_i - q^*_i)(v_i - p^*) \le 0$. Consequently, it satisfies that $u'_i \le u^*_i$ for all $x'_i = (v'_i, w'_i)$ with $w'_i \le w_i$. □

Appendix B: Comparisons between characteristics of discrete and continuum-mass hybrid mechanisms

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In the following table, $Ef(r, m_1)$, $Eq(r, m_1)$, and $Re(r, m_1)$ represent the computed efficiency, equality, and revenue of the discrete hybrid mechanism

$H(r,m_1)$, respectively. $Ef(r,\frac{m_1}{n})$, $Eq(r,\frac{m_1}{n})$, and $Re(r,\frac{m_1}{n})$ represent the computed efficiency, equality, and revenue of the continuum-mass hybrid mechanism $H(r,\frac{m_1}{n})$, respectively. $Diff_{Ef}, Diff_{Eq}$, and $Diff_{Re}$ denote the relative difference between computed characteristics of $H(r,\frac{m_1}{n})$ and those of $H(r,m_1)$. Buyers' types are assumed to be uniformly distributed.

r	m_1	$Ef(r,m_1)$	$Eq(r,m_1)$	$Re(r,m_1)$	$Ef(r,\frac{m_1}{n})$	$Eq(r,\frac{m_1}{n})$	$Re(r,\frac{m_1}{n})$	Dif_{Ef}	Dif_{Eq}	Dif_{Re}
0	0	5000.09	0.999919	0.00	5000.00	1.000000	0.00	-0.002%	0.008%	0.000%
0	1000	5424.18	0.922825	848.90	5424.46	0.922826	848.92	0.005%	0.000%	0.002%
0	2000	5817.85	0.849801	1635.26	5817.64	0.849820	1635.29	-0.004%	0.002%	0.002%
0	3000	6189.43	0.779192	2379.07	6189.83	0.779206	2379.67	0.007%	0.002%	0.025%
0	4000	6545.41	0.710296	3090.01	6545.19	0.710277	3090.38	-0.003%	-0.003%	0.012%
0	5000	6886.16	0.642631	3771.87	6886.14	0.642626	3772.28	0.000%	-0.001%	0.011%
0	6000	7214.24	0.575980	4428.40	7214.30	0.575985	4428.60	0.001%	0.001%	0.005%
0	7000	7530.64	0.510111	5061.65	7530.85	0.510159	5061.69	0.003%	0.009%	0.001%
0	8000	7836.68	0.445099	5673.11	7836.67	0.444999	5673.34	0.000%	-0.022%	0.004%
0	9000	8132.69	0.380413	6264.49	8132.49	0.380388	6264.97	-0.002%	-0.007%	0.008%
0	10000	8418.61	0.316301	6838.14	8418.86	0.316228	6837.72	0.003%	-0.023%	-0.006%
1000	0	5499.90	0.899912	1000.00	5500.00	0.900000	1000.00	0.002%	0.010%	0.000%
1000	1000	5871.63	0.834156	1742.38	5871.21	0.834110	1742.43	-0.007%	-0.006%	0.003%
1000	2000	6212.79	0.771880	2425.41	6212.83	0.771871	2425.66	0.001%	-0.001%	0.010%
1000	3000	6533.99	0.711749	3068.22	6534.25	0.711713	3068.51	0.004%	-0.005%	0.009%
1000	4000	6839.48	0.653076	3678.91	6839.42	0.652990	3678.84	-0.001%	-0.013%	-0.002%
1000	5000	7131.03	0.595284	4260.90	7130.63	0.595330	4261.25	-0.006%	0.008%	0.008%
1000	6000	7409.43	0.538526	4819.10	7409.40	0.538487	4818.80	0.000%	-0.007%	-0.006%
1000	7000	7676.89	0.482306	5353.47	7676.86	0.482279	5353.71	0.000%	-0.006%	0.004%
1000	8000	7933.53	0.426662	5867.45	7933.83	0.426570	5867.69	0.004%	-0.022%	0.004%
1000	9000	8180.71	0.371219	6361.66	8180.99	0.371250	6361.98	0.003%	0.008%	0.005%
1000	10000	8418.94	0.316186	6837.71	8418.86	0.316228	6837.72	-0.001%	0.013%	0.000%
2000	0	5999.89	0.799943	2000.00	6000.00	0.800000	2000.00	0.002%	0.007%	0.000%
2000	1000	6317.02	0.745589	2633.97	6317.03	0.745652	2634.06	0.000%	0.008%	0.004%
2000	2000	6606.58	0.694336	3213.06	6606.67	0.694322	3213.34	0.001%	-0.002%	0.009%
2000	3000	6876.96	0.644733	3754.36	6877.19	0.644694	3754.38	0.003%	-0.006%	0.000%
2000	4000	7132.44	0.596267	4264.17	7132.19	0.596207	4264.37	-0.004%	-0.010%	0.005%
2000	5000	7374.18	0.548458	4747.26	7373.78	0.548529	4747.55	-0.005%	0.013%	0.006%
2000	6000	7602.87	0.501385	5206.70	7603.37	0.501441	5206.75	0.007%	0.011%	0.001%
2000	7000	7821.66	0.454824	5643.09	7821.99	0.454780	5643.99	0.004%	-0.010%	0.016%
2000	8000	8029.95	0.408480	6060.71	8030.40	0.408422	6060.81	0.006%	-0.014%	0.002%
2000	9000	8229.37	0.362164	6458.42	8229.20	0.362266	6458.40	-0.002%	0.028%	0.000%
2000	10000	8419.63	0.316188	6837.58	8418.86	0.316228	6837.72	-0.009%	0.013%	0.002%
3000	0	6499.60	0.700002	3000.00	6500.00	0.700000	3000.00	0.006%	0.000%	0.000%
3000	1000	6761.16	0.657535	3522.87	6761.44	0.657516	3522.88	0.004%	-0.003%	0.000%
3000	2000	6998.74	0.617236	3996.91	6998.49	0.617271	3996.99	-0.003%	0.006%	0.002%
3000	3000	7217.76	0.578274	4435.47	7217.90	0.578270	4435.79	0.002%	-0.001%	0.007%
3000	4000	7422.61	0.540086	4845.44	7422.77	0.540053	4845.54	0.002%	-0.006%	0.002%
3000	5000	7614.82	0.502336	5229.79	7614.96	0.502348	5229.92	0.002%	0.002%	0.003%
3000	6000	7795.89	0.464964	5590.76	7795.71	0.464965	5591.42	-0.002%	0.000%	0.012%
3000	7000	7965.68	0.427845	5931.38	7965.89	0.427763	5931.78	0.003%	-0.019%	0.007%
3000	8000	8125.78	0.390589	6252.51	8126.16	0.390632	6252.33	0.005%	0.011%	-0.003%
3000	9000	8277.21	0.353557	6553.84	8277.03	0.353480	6554.05	-0.002%	-0.022%	0.003%
3000	10000	8418.31	0.316304	6837.30	8418.86	0.316228	6837.72	0.007%	-0.024%	0.006%
4000	0	6999.62	0.600030	4000.00	7000.00	0.600000	4000.00	0.005%	-0.005%	0.000%
4000	1000	7203.54	0.569886	4407.22	7203.59	0.569752	4407.19	0.001%	-0.024%	-0.001%
4000	2000	7386.38	0.540814	4773.78	7387.02	0.540809	4774.03	0.009%	-0.001%	0.005%
4000	3000	7554.86	0.512541	5109.86	7554.96	0.512552	5109.92	0.001%	0.002%	0.001%
4000	4000	7709.42	0.484655	5419.61	7709.80	0.484657	5419.60	0.005%	0.001%	0.000%
4000	5000	7852.69	0.456924	5705.78	7852.98	0.456919	5705.96	0.004%	-0.001%	0.003%
4000	6000	7985.69	0.429217	5970.69	7985.45	0.429189	5970.90	-0.003%	-0.007%	0.004%
4000	7000	8107.78	0.401402	6215.46	8107.87	0.401348	6215.74	0.001%	-0.013%	0.005%

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r	m_1	$Ef(r,m_1)$	$Eq(r,m_1)$	$Re(r,m_1)$	$Ef(r,\frac{m_1}{n})$	$Eq(r,\frac{m_1}{n})$	$Re(r,\frac{m_1}{n})$	Dif_{Ef}	Dif_{Eq}	Dif_{Re}
4000	8000	8220.54	0.373338	6440.94	8220.71	0.373298	6441.41	0.002%	-0.011%	0.007%
4000	9000	8324.41	0.344948	6648.20	8324.29	0.344951	6648.59	-0.001%	0.001%	0.006%
4000	10000	8418.80	0.316289	6836.96	8418.86	0.316228	6837.72	0.001%	-0.019%	0.011%
5000	0	7500.24	0.500027	5000.00	7500.00	0.500000	5000.00	-0.003%	-0.005%	0.000%
5000	1000	7641.92	0.482241	5283.64	7641.81	0.482274	5283.62	-0.002%	0.007%	-0.001%
5000	2000	7769.57	0.464808	5538.97	7769.50	0.464848	5539.00	-0.001%	0.009%	0.001%
5000	3000	7885.03	0.447403	5770.18	7885.20	0.447472	5770.41	0.002%	0.015%	0.004%
5000	4000	7990.44	0.429969	5980.25	7990.14	0.429981	5980.27	-0.004%	0.003%	0.000%
5000	5000	8084.78	0.412279	6169.83	8085.04	0.412244	6170.07	0.003%	-0.008%	0.004%
5000	6000	8169.94	0.394298	6340.66	8170.36	0.394154	6340.71	0.005%	-0.037%	0.001%
5000	7000	8246.38	0.375618	6492.96	8246.35	0.375608	6492.70	0.000%	-0.003%	-0.004%
5000	8000	8312.57	0.356517	6625.28	8313.12	0.356508	6626.25	0.007%	-0.002%	0.015%
5000	9000	8371.07	0.336760	6741.35	8370.67	0.336750	6741.33	-0.005%	-0.003%	0.000%
5000	10000	8418.77	0.316294	6837.97	8418.86	0.316228	6837.72	0.001%	-0.021%	-0.004%
6000	0	7999.87	0.400051	6000.00	8000.00	0.400000	6000.00	0.002%	-0.013%	0.000%
6000	1000	8071.84	0.394220	6144.34	8072.18	0.394226	6144.35	0.004%	0.002%	0.000%
6000	2000	8138.74	0.388067	6277.36	8138.63	0.388117	6277.27	-0.001%	0.013%	-0.001%
6000	3000	8198.62	0.381588	6398.39	8199.19	0.381613	6398.39	0.007%	0.006%	0.000%
6000	4000	8253.26	0.374600	6507.25	8253.59	0.374641	6507.18	0.004%	0.011%	-0.001%
6000	5000	8301.38	0.367143	6602.55	8301.44	0.367115	6602.89	0.001%	-0.008%	0.005%
6000	6000	8342.14	0.359055	6684.39	8342.26	0.358929	6684.52	0.002%	-0.035%	0.002%
6000	7000	8375.27	0.349958	6750.31	8375.39	0.349947	6750.79	0.001%	-0.003%	0.007%
6000	8000	8400.12	0.339994	6800.22	8400.00	0.340000	6800.00	-0.001%	0.002%	-0.003%
6000	9000	8415.14	0.328874	6830.14	8414.98	0.328861	6829.95	-0.002%	-0.004%	-0.003%
6000	10000	8418.90	0.316294	6837.51	8418.86	0.316228	6837.72	0.000%	-0.021%	0.003%
7000	0	7650.92	0.300057	6297.26	7650.00	0.300000	6300.00	-0.012%	-0.019%	0.044%
7000	1000	7652.60	0.299894	6299.86	7650.00	0.300000	6300.00	-0.034%	0.036%	0.002%
7000	2000	7648.85	0.300241	6303.28	7650.00	0.300000	6300.00	0.015%	-0.080%	-0.052%
7000	3000	7649.41	0.300138	6302.06	7650.00	0.300000	6300.00	0.008%	-0.046%	-0.033%
7000	4000	7656.82	0.299784	6297.43	7650.00	0.300000	6300.00	-0.089%	0.072%	0.041%
7000	5000	7648.99	0.299804	6297.59	7650.00	0.300000	6300.00	0.013%	0.065%	0.038%
7000	6000	7651.31	0.300165	6300.64	7650.00	0.300000	6300.00	-0.017%	-0.055%	-0.010%
7000	7000	7652.21	0.299963	6303.34	7650.00	0.300000	6300.00	-0.029%	0.012%	-0.053%
7000	8000	7647.05	0.300182	6296.54	7650.00	0.300000	6300.00	0.039%	-0.060%	0.055%
7000	9000	7652.63	0.299984	6301.19	7650.00	0.300000	6300.00	-0.034%	0.005%	-0.019%
7000	10000	7649.25	0.299956	6301.77	7650.00	0.300000	6300.00	0.010%	0.015%	-0.028%
8000	0	3598.52	0.199858	3201.02	3600.00	0.200000	3200.00	0.041%	0.071%	-0.032%
8000	1000	3598.64	0.200114	3200.66	3600.00	0.200000	3200.00	0.038%	-0.057%	-0.021%
8000	2000	3600.16	0.200018	3200.65	3600.00	0.200000	3200.00	-0.004%	-0.009%	-0.020%
8000	3000	3602.56	0.200099	3200.79	3600.00	0.200000	3200.00	-0.071%	-0.050%	-0.025%
8000	4000	3599.96	0.200021	3201.02	3600.00	0.200000	3200.00	0.001%	-0.011%	-0.032%
8000	5000	3599.36	0.199955	3198.02	3600.00	0.200000	3200.00	0.018%	0.022%	0.062%
8000	6000	3599.75	0.200281	3200.78	3600.00	0.200000	3200.00	0.007%	-0.140%	-0.024%
8000	7000	3603.34	0.199869	3200.63	3600.00	0.200000	3200.00	-0.093%	0.066%	-0.020%
8000	8000	3600.70	0.200214	3200.19	3600.00	0.200000	3200.00	-0.019%	-0.107%	-0.006%
8000	9000	3599.05	0.200057	3198.74	3600.00	0.200000	3200.00	0.026%	-0.029%	0.039%
8000	10000	3603.64	0.200052	3201.47	3600.00	0.200000	3200.00	-0.101%	-0.026%	-0.046%
9000	0	949.35	0.100101	900.01	950.00	0.100000	900.00	0.068%	-0.101%	-0.001%
9000	1000	948.80	0.099910	898.69	950.00	0.100000	900.00	0.126%	0.090%	0.146%
9000	2000	949.04	0.099885	899.51	950.00	0.100000	900.00	0.101%	0.115%	0.055%
9000	3000	949.61	0.100083	900.76	950.00	0.100000	900.00	0.041%	-0.083%	-0.084%
9000	4000	949.13	0.099985	901.41	950.00	0.100000	900.00	0.092%	0.015%	-0.156%
9000	5000	950.95	0.099973	898.67	950.00	0.100000	900.00	-0.100%	0.027%	0.148%
9000	6000	949.24	0.099976	900.78	950.00	0.100000	900.00	0.080%	0.024%	-0.086%
9000	7000	950.34	0.100060	900.14	950.00	0.100000	900.00	-0.035%	-0.060%	-0.015%
9000	8000	950.21	0.100012	898.89	950.00	0.100000	900.00	-0.022%	-0.012%	0.124%
9000	9000	949.76	0.100040	900.30	950.00	0.100000	900.00	0.025%	-0.040%	-0.033%
9000	10000	950.03	0.099958	899.95	950.00	0.100000	900.00	-0.003%	0.042%	0.006%
10000	0	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	1000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	2000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	3000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	4000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	5000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	6000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%

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r	m_1	$Ef(r,m_1)$	$Eq(r,m_1)$	$Re(r,m_1)$	$Ef(r,\frac{m_1}{n})$	$Eq(r,\frac{m_1}{n})$	$Re(r,\frac{m_1}{n})$	Dif_{Ef}	Dif_{Eq}	Dif_{Re}
10000	7000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	8000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	9000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	10000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%

The following table presents the similar results of comparisons with exponential distributed buyers' types.

r	m_1	$Ef(r,m_1)$	$Eq(r,m_1)$	$Re(r,m_1)$	$Ef(r,\frac{m_1}{n})$	$Eq(r,\frac{m_1}{n})$	$Re(r,\frac{m_1}{n})$	Dif_{Ef}	Dif_{Eq}	Dif_{Re}
0	0	5000.08	0.999911	0.00	5000.00	1.000000	0.00	-0.002%	0.009%	0.000%
0	1000	6095.69	0.919248	1095.79	6096.47	0.919235	1096.47	0.013%	-0.001%	0.062%
0	2000	6871.71	0.844542	1872.62	6872.78	0.844556	1872.78	0.015%	0.002%	0.008%
0	3000	7530.19	0.773169	2531.26	7531.47	0.773132	2531.47	0.017%	-0.005%	0.009%
0	4000	8115.84	0.703995	3114.96	8115.04	0.703997	3115.04	-0.010%	0.000%	0.003%
0	5000	8642.34	0.636615	3642.77	8643.46	0.636611	3643.46	0.013%	-0.001%	0.019%
0	6000	9127.22	0.570680	4127.95	9128.37	0.570617	4128.37	0.013%	-0.011%	0.010%
0	7000	9578.16	0.505831	4576.73	9577.41	0.505759	4577.41	-0.008%	-0.014%	0.015%
0	8000	9994.27	0.441793	4996.25	9995.95	0.441840	4995.95	0.017%	0.011%	-0.006%
0	9000	10385.86	0.378719	5386.20	10387.94	0.378705	5387.94	0.020%	-0.004%	0.032%
0	10000	10754.89	0.316171	5756.09	10756.46	0.316228	5756.46	0.015%	0.018%	0.006%
1000	0	6000.47	0.818689	1000.00	6000.00	0.818731	1000.00	-0.008%	0.005%	0.000%
1000	1000	6974.12	0.758125	1973.85	6974.41	0.758089	1974.41	0.004%	-0.005%	0.028%
1000	2000	7641.02	0.703054	2640.65	7640.92	0.702958	2640.92	-0.001%	-0.014%	0.011%
1000	3000	8196.06	0.650569	3195.02	8195.20	0.650593	3195.20	-0.010%	0.004%	0.006%
1000	4000	8675.03	0.600140	3677.51	8678.01	0.600079	3678.01	0.034%	-0.010%	0.013%
1000	5000	9109.33	0.550960	4107.84	9108.41	0.550911	4108.41	-0.010%	-0.009%	0.014%
1000	6000	9497.30	0.502751	4495.95	9497.47	0.502756	4497.47	0.002%	0.001%	0.034%
1000	7000	9853.30	0.455422	4852.26	9852.38	0.455375	4852.38	-0.009%	-0.010%	0.003%
1000	8000	10176.00	0.408488	5178.09	10178.20	0.408586	5178.20	0.022%	0.024%	0.002%
1000	9000	10477.03	0.362303	5478.21	10478.61	0.362243	5478.61	0.015%	-0.017%	0.007%
1000	10000	10756.49	0.316201	5756.56	10756.46	0.316228	5756.46	0.000%	0.009%	-0.002%
2000	0	7000.75	0.670306	2000.00	7000.00	0.670320	2000.00	-0.011%	0.002%	0.000%
2000	1000	7842.75	0.626892	2841.92	7842.73	0.626900	2842.73	0.000%	0.001%	0.029%
2000	2000	8394.90	0.588345	3395.69	8396.15	0.588241	3396.15	0.015%	-0.018%	0.013%
2000	3000	8845.23	0.551788	3844.99	8845.06	0.551758	3845.06	-0.002%	-0.005%	0.002%
2000	4000	9228.92	0.516614	4227.71	9227.57	0.516621	4227.57	-0.015%	0.001%	-0.003%
2000	5000	9560.97	0.482321	4561.45	9561.36	0.482371	4561.36	0.004%	0.010%	-0.002%
2000	6000	9854.51	0.448767	4855.94	9856.57	0.448708	4856.57	0.021%	-0.013%	0.013%
2000	7000	10119.11	0.415365	5119.15	10119.74	0.415415	5119.74	0.006%	0.012%	0.012%
2000	8000	10353.08	0.382417	5356.11	10355.40	0.382326	5355.40	0.022%	-0.024%	-0.013%
2000	9000	10566.56	0.349254	5566.33	10566.83	0.349302	5566.83	0.003%	0.014%	0.009%
2000	10000	10757.23	0.316368	5756.03	10756.46	0.316228	5756.46	-0.007%	-0.044%	0.008%
3000	0	8000.47	0.548800	3000.00	8000.00	0.548812	3000.00	-0.006%	0.002%	0.000%
3000	1000	8694.65	0.520237	3695.53	8695.69	0.520324	3695.69	0.012%	0.017%	0.004%
3000	2000	9131.23	0.495674	4131.13	9131.13	0.495618	4131.13	-0.001%	-0.011%	0.000%
3000	3000	9472.23	0.472417	4473.35	9473.57	0.472399	4473.57	0.014%	-0.004%	0.005%
3000	4000	9757.11	0.449960	4756.34	9756.90	0.449962	4756.90	-0.002%	0.000%	0.012%
3000	5000	9997.23	0.427945	4996.87	9996.57	0.427917	4996.57	-0.007%	-0.006%	-0.006%
3000	6000	10200.98	0.406082	5201.22	10201.29	0.406006	5201.29	0.003%	-0.019%	0.001%
3000	7000	10372.08	0.384019	5375.24	10376.52	0.384033	5376.52	0.043%	0.004%	0.024%
3000	8000	10526.88	0.361778	5525.08	10525.91	0.361838	5525.91	-0.009%	0.016%	0.015%
3000	9000	10650.27	0.339309	5650.19	10651.97	0.339279	5651.97	0.016%	-0.009%	0.032%
3000	10000	10754.81	0.316292	5757.90	10756.46	0.316228	5756.46	0.015%	-0.020%	-0.025%
4000	0	8998.27	0.449324	4000.00	9000.00	0.449329	4000.00	0.019%	0.001%	0.000%
4000	1000	9520.69	0.433872	4520.68	9521.14	0.433884	4521.14	0.005%	0.003%	0.010%
4000	2000	9830.16	0.420935	4830.16	9830.14	0.420936	4830.14	0.000%	0.000%	-0.001%
4000	3000	10066.34	0.408699	5064.90	10064.33	0.408726	5064.33	-0.020%	0.007%	-0.011%
4000	4000	10248.23	0.396720	5249.90	10250.66	0.396737	5250.66	0.024%	0.004%	0.015%
4000	5000	10402.01	0.384565	5401.69	10400.95	0.384673	5400.95	-0.010%	0.028%	-0.014%
4000	6000	10521.75	0.372313	5521.01	10521.48	0.372316	5521.48	-0.003%	0.001%	0.009%
4000	7000	10615.56	0.359465	5615.62	10615.84	0.359479	5615.84	0.003%	0.004%	0.004%
4000	8000	10686.40	0.346031	5685.56	10686.00	0.345981	5686.00	-0.004%	-0.014%	0.008%
4000	9000	10732.96	0.331623	5732.86	10732.85	0.331634	5732.85	-0.001%	0.003%	0.000%

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r	m_1	$Ef(r,m_1)$	$Eq(r,m_1)$	$Re(r,m_1)$	$Ef(r,\frac{m_1}{n})$	$Eq(r,\frac{m_1}{n})$	$Re(r,\frac{m_1}{n})$	Dif_{Ef}	Dif_{Eq}	Dif_{Re}
4000	10000	10757.08	0.316275	5756.14	10756.46	0.316228	5756.46	-0.006%	-0.015%	0.006%
5000	0	9998.00	0.367911	5000.00	10000.00	0.367879	5000.00	0.020%	-0.009%	0.000%
5000	1000	10283.92	0.363458	5286.15	10286.47	0.363356	5286.47	0.025%	-0.028%	0.006%
5000	2000	10448.00	0.359801	5448.09	10448.41	0.359734	5448.41	0.004%	-0.019%	0.006%
5000	3000	10566.30	0.356250	5567.84	10567.65	0.356215	5567.65	0.013%	-0.010%	-0.003%
5000	4000	10657.60	0.352576	5658.53	10659.20	0.352565	5659.20	0.015%	-0.003%	0.012%
5000	5000	10728.43	0.348659	5729.96	10728.94	0.348616	5728.94	0.005%	-0.012%	-0.018%
5000	6000	10778.11	0.344223	5779.88	10779.13	0.344200	5779.13	0.009%	-0.007%	-0.013%
5000	7000	10811.47	0.339129	5809.86	10809.90	0.339109	5809.90	-0.015%	-0.006%	0.001%
5000	8000	10818.38	0.333138	5819.40	10819.53	0.333064	5819.53	0.011%	-0.022%	0.002%
5000	9000	10802.58	0.325599	5802.64	10804.10	0.325652	5804.10	0.014%	0.016%	0.025%
5000	10000	10755.42	0.316222	5754.93	10756.46	0.316228	5756.46	0.010%	0.002%	0.027%
6000	0	9978.27	0.301103	5443.72	9978.97	0.301194	5443.08	0.007%	0.030%	-0.012%
6000	1000	9978.50	0.301119	5443.73	9978.97	0.301194	5443.08	0.005%	0.025%	-0.012%
6000	2000	9980.90	0.301258	5442.88	9978.97	0.301194	5443.08	-0.019%	-0.021%	0.004%
6000	3000	9982.01	0.301352	5444.84	9978.97	0.301194	5443.08	-0.030%	-0.052%	-0.032%
6000	4000	9984.73	0.301194	5441.91	9978.97	0.301194	5443.08	-0.058%	0.000%	0.021%
6000	5000	9980.83	0.300867	5442.83	9978.97	0.301194	5443.08	-0.019%	0.109%	0.005%
6000	6000	9976.23	0.301284	5439.83	9978.97	0.301194	5443.08	0.028%	-0.030%	0.060%
6000	7000	9985.65	0.301350	5442.63	9978.97	0.301194	5443.08	-0.067%	-0.052%	0.008%
6000	8000	9977.38	0.301247	5441.49	9978.97	0.301194	5443.08	0.016%	-0.018%	0.029%
6000	9000	9978.29	0.301443	5444.26	9978.97	0.301194	5443.08	0.007%	-0.083%	-0.022%
6000	10000	9976.47	0.300933	5439.76	9978.97	0.301194	5443.08	0.025%	0.087%	0.061%
7000	0	7293.20	0.246530	4253.12	7297.21	0.246597	4256.70	0.055%	0.027%	0.084%
7000	1000	7298.56	0.246488	4257.01	7297.21	0.246597	4256.70	-0.018%	0.044%	-0.007%
7000	2000	7297.00	0.246669	4257.72	7297.21	0.246597	4256.70	0.003%	-0.029%	-0.024%
7000	3000	7298.33	0.246880	4255.08	7297.21	0.246597	4256.70	-0.015%	-0.115%	0.038%
7000	4000	7295.60	0.246347	4254.06	7297.21	0.246597	4256.70	0.022%	0.101%	0.062%
7000	5000	7296.20	0.246760	4257.12	7297.21	0.246597	4256.70	0.014%	-0.066%	-0.010%
7000	6000	7295.83	0.246751	4256.17	7297.21	0.246597	4256.70	0.019%	-0.062%	0.013%
7000	7000	7297.36	0.246464	4256.03	7297.21	0.246597	4256.70	-0.002%	0.054%	0.016%
7000	8000	7292.28	0.246641	4254.38	7297.21	0.246597	4256.70	0.068%	-0.018%	0.055%
7000	9000	7299.53	0.246548	4256.35	7297.21	0.246597	4256.70	-0.032%	0.020%	0.008%
7000	10000	7297.47	0.246605	4257.46	7297.21	0.246597	4256.70	-0.004%	-0.003%	-0.018%
8000	0	5300.57	0.201915	3257.00	5299.09	0.201897	3260.98	-0.028%	-0.009%	0.122%
8000	1000	5304.05	0.201816	3260.26	5299.09	0.201897	3260.98	-0.094%	0.040%	0.022%
8000	2000	5299.36	0.202042	3264.41	5299.09	0.201897	3260.98	-0.005%	-0.072%	-0.105%
8000	3000	5298.77	0.201896	3261.42	5299.09	0.201897	3260.98	0.006%	0.000%	-0.014%
8000	4000	5298.19	0.201912	3260.72	5299.09	0.201897	3260.98	0.017%	-0.008%	0.008%
8000	5000	5301.89	0.201865	3259.62	5299.09	0.201897	3260.98	-0.053%	0.016%	0.042%
8000	6000	5298.29	0.201995	3261.30	5299.09	0.201897	3260.98	0.015%	-0.049%	-0.010%
8000	7000	5294.70	0.201829	3259.46	5299.09	0.201897	3260.98	0.083%	0.033%	0.047%
8000	8000	5307.03	0.201951	3260.23	5299.09	0.201897	3260.98	-0.150%	-0.027%	0.023%
8000	9000	5298.80	0.201828	3259.75	5299.09	0.201897	3260.98	0.005%	0.034%	0.038%
8000	10000	5297.73	0.201940	3262.21	5299.09	0.201897	3260.98	0.026%	-0.021%	-0.038%
9000	0	3821.88	0.165397	2458.53	3825.32	0.165299	2459.14	0.090%	-0.059%	0.024%
9000	1000	3824.07	0.165210	2460.75	3825.32	0.165299	2459.14	0.033%	0.054%	-0.066%
9000	2000	3828.63	0.165300	2461.19	3825.32	0.165299	2459.14	-0.086%	-0.001%	-0.083%
9000	3000	3827.14	0.165284	2458.33	3825.32	0.165299	2459.14	-0.048%	0.009%	0.033%
9000	4000	3825.61	0.165300	2461.25	3825.32	0.165299	2459.14	-0.007%	-0.001%	-0.086%
9000	5000	3820.51	0.165254	2459.44	3825.32	0.165299	2459.14	0.126%	0.027%	-0.012%
9000	6000	3829.83	0.165284	2460.88	3825.32	0.165299	2459.14	-0.118%	0.009%	-0.071%
9000	7000	3827.34	0.165300	2460.72	3825.32	0.165299	2459.14	-0.053%	-0.001%	-0.064%
9000	8000	3821.39	0.165254	2459.83	3825.32	0.165299	2459.14	0.103%	0.027%	-0.028%
9000	9000	3823.16	0.165345	2457.97	3825.32	0.165299	2459.14	0.057%	-0.028%	0.047%
9000	10000	3825.98	0.165284	2457.09	3825.32	0.165299	2459.14	-0.017%	0.009%	0.083%
10000	0	2750.49	0.135363	1832.41	2747.35	0.135335	1831.56	-0.114%	-0.020%	-0.046%
10000	1000	2748.85	0.135431	1831.23	2747.35	0.135335	1831.56	-0.055%	-0.070%	0.018%
10000	2000	2742.58	0.135334	1832.21	2747.35	0.135335	1831.56	0.174%	0.001%	-0.035%
10000	3000	2746.74	0.135367	1829.56	2747.35	0.135335	1831.56	0.022%	-0.024%	0.109%
10000	4000	2748.50	0.135264	1832.44	2747.35	0.135335	1831.56	-0.042%	0.053%	-0.048%
10000	5000	2743.25	0.135337	1833.26	2747.35	0.135335	1831.56	0.149%	-0.001%	-0.093%
10000	6000	2748.85	0.135345	1832.11	2747.35	0.135335	1831.56	-0.055%	-0.007%	-0.030%
10000	7000	2748.14	0.135349	1834.10	2747.35	0.135335	1831.56	-0.029%	-0.010%	-0.138%
10000	8000	2745.06	0.135242	1832.60	2747.35	0.135335	1831.56	0.083%	0.069%	-0.057%

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r	m_1	$Ef(r, m_1)$	$Eq(r, m_1)$	$Re(r, m_1)$	$Ef(r, \frac{m_1}{n})$	$Eq(r, \frac{m_1}{n})$	$Re(r, \frac{m_1}{n})$	Dif_{Ef}	Dif_{Eq}	Dif_{Re}
10000	9000	2747.76	0.135350	1832.87	2747.35	0.135335	1831.56	-0.015%	-0.011%	-0.071%
10000	10000	2746.79	0.135289	1830.36	2747.35	0.135335	1831.56	0.020%	0.034%	0.066%

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Journal of Mechanism and Institution Design
ISSN: 2399-844X (Print), 2399-8458 (Online)
DOI:10.22574/jmid.2019.11.003

PARETO OPTIMAL COALITIONS OF FIXED SIZE

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ABSTRACT

We tackle the problem of partitioning players into groups of fixed size, such as allocating eligible students to shared dormitory rooms. Each student submits preferences over the other individual students. We study several settings, which differ in the size of the rooms to be filled, the orderedness or completeness of the preferences, and the way of calculating the value of a coalition—based on the best or worst roommate in the coalition. In all cases, we determine the complexity of deciding the existence, and then finding a Pareto optimal assignment, and the complexity of verifying Pareto optimality for a given assignment.

Keywords: Coalition formation, Pareto optimality, complexity.

JEL Classification Numbers: C70, D47.

All authors declare there are no conflicts of interest. We thank the Cooperation of Excellences Grant (KEP-6/2019), by the Ministry of Human Resources under its New National Excellence Programme (ÚNKP-19-4-BME-335), the Hungarian Academy of Sciences under its Momentum Programme (LP2016-3/2019), its János Bolyai Research Fellowship, the National Research, Development and Innovation Fund (TUDFO/51757/2019-ITM, Thematic Excellence Program), and OTKA grant K128611. Any errors are our own.

1. INTRODUCTION

The ubiquitous nature of coalition formation has stimulated researchers to study the behavior of individuals forming groups (Dreze & Greenberg, 1980; Gamson, 1961; Kahan & Rapoport, 2014; Kelso Jr & Crawford, 1982; Shehory & Kraus, 1998). A large portion of the game-theoretical studies focuses on individuals who have ordinal preferences over the possible outcomes (Banerjee et al., 2001; Bogomolnaia & Jackson, 2002; Cechlárová & Hajduková, 2004a,b; Peters & Elkind, 2015).

Our setting involves n players who need to be partitioned into coalitions. For convenience, we talk about assigning each player to a room. We assume the rooms to have no specific feature besides their capacity. To ensure a feasible partition in the outcome, we assume that the total capacity of rooms adds up to n, and a feasible assignment fills each room to its capacity. Having once entered the scheme, players have no option to opt out. Each player submits her preference list over the other players. Four features of the problem define various settings, each of which is realistic and will be investigated by us.

(i) Preference lists might be *complete* or *incomplete*, the latter term meaning that players have the right to declare some of the other players unacceptable as a roommate. No player can be put in the same room with an unacceptable roommate.

(ii) Players might have *strictly* or *weakly ordered* lists.

(iii) Players might compare two coalitions based on their *best* or *worst* assigned roommate.

(iv) The rooms might all accommodate 3 players, or they might have a predefined *capacity* each, which we denote by $r_1, r_2, \ldots, r_k$ for each of the k rooms.

The optimality principle we are focusing on is *Pareto optimality*. A room assignment, or a set of coalitions, is Pareto optimal if there is no other assignment in which at least one player is better off, and no player is worse off than in the first assignment. The comparison here is defined based on point (iii) above. We shorten the term 'Pareto optimal assignment' to POA. Our goal is to study all 2^4 combinations of the above four features, and for each of them, determine the complexity of the following three problems:

(1) verifying whether a given feasible assignment is a POA;

(2) checking whether a POA exists;

(3) finding a POA.

1.1. Related literature

Pareto optimality in coalition formation has a rich literature. We start with an overview on coalition formation viewed as a hedonic game. Then we review the settings in which the coalition size matters.

1.1.1. Hedonic games

Coalition formation under preferences can be seen as a hedonic game (Aziz et al., 2011; Banerjee et al., 2001; Bogomolnaia & Jackson, 2002). In such a game, players have preferences over the possible coalitions they can be part of, and coalitions can be of any size—notice that these two basic features differ strikingly from our setting. Pareto optimal coalition formation as a hedonic game is extensively studied by Aziz et al. (2013). They analyze two restricted variants of hedonic games that are closely related to our setting. Both of them operate under the assumption that the coalition size is arbitrary, but they both derive players' preferences on coalitions from a preference list on individual players. They show that if the preference lists are incomplete, and preferences are based on the best roommate, then verifying Pareto optimality for a given assignment is coNP-complete, and computing a POA is NP-hard. For complete lists and the same preferences, the grand coalition is a trivial optimal solution, since every player has their first-choice roommate in the sole room. Aziz et al. (2013) also show that if preferences are based on the worst roommate, then both verifying Pareto optimality for a given assignment, and computing a POA can be done in polynomial time.

1.1.2. 2-person rooms

Some of the literature concentrates on each coalition being of size 1 or 2. In this setting, a player can compare two coalitions simply based on the rank of the sole roommate (if any exists), so our point (iii) does not apply here. Using Morrill's algorithm (Morrill, 2010), Aziz et al. (2013) show that even if preferences contain ties, both verifying Pareto optimality for a given matching, and calculating a POA are solvable in polynomial time. Their results are valid for complete and incomplete lists as well. Abraham & Manlove (2004) consider Pareto optimal matchings as a means of coping with instances of the stable roommates problem with strict lists, which do not admit a stable matching. They show that while a maximum size POA is easy to find, finding a minimum size POA is NP-hard.

1.1.3. 3-person rooms

For the setting in which a room can accommodate up to 3 players, two versions of the problem have been studied. A *three-cyclic game* is a hedonic game in which the set of players is divided into men, women, and dogs and the only kind of acceptable coalitions are man-woman-dog triplets (Knuth, 1976; Ng & Hirschberg, 1991). Furthermore, men only care about women, women only care about dogs and dogs only care about men. Computing a POA is known to be NP-hard for these games, while the corresponding verification problem is coNP-complete, even for strict preferences (Aziz et al., 2013). In *room-roommate games*, a set of players act as rooms, and these have no preferences whatsoever. The ordinary players, on the other hand, have a preference list over all possible roommate-room pairs they find acceptable. A triplet is feasible if exactly one player in it plays a room. If preferences are strict, then a POA can be computed in polynomial time, but the problem becomes NP-hard as soon as ties are introduced, even if all lists are complete (Aziz et al., 2013). Just as in the previous problem, the verification version is coNP-complete, even for strict preferences (Aziz et al., 2013).

1.1.4. Preferences depending on the room size

Anonymous hedonic games (Ballester, 2004) are a subclass of hedonic games in which the players' preferences over coalitions only depend on coalition sizes. Both verification and finding a POA are hard in such games (Aziz et al., 2013). Darmann (2018) studies a group activity selection model in which players have preferences not only on the activity, but also on the number of participants in their coalition. He provides an efficient algorithm to find a POA, if each player wants to share an activity with as many, or as few players as possible.

As we have seen, a number of papers investigate the complexity of finding a POA under various settings, such as limited coalition size or preferences over players rather than over coalitions. However, there is no work on the combination of these two. In our setting, homogeneous players *rank each other*, and form coalitions of an arbitrary, but *fixed* size.

1.2. Our contribution

We tackle the problems of verifying Pareto optimality, deciding the existence of POA, and finding a POA for a set of fixed coalition sizes. We distinguish 2^4 cases, based on the completeness and the orderedness of preferences, the way of comparing two coalitions by a player, and the room sizes. Our findings are summarized below and in Figure 1.

- Verification is coNP-complete in all cases. We show this in Section 3 by two reductions from triangle cover problems.
- If lists are incomplete, then deciding whether a feasible assignment exists is already NP-complete in all cases. On the other hand, if a feasible assignment does exist, then an optimal one exists as well. These are due to a simple NP-completeness reduction and a monotonicity argument, which can be found in Section 4.
- For complete lists, a POA is bound to exist. In 3 out of the 16 cases, serial dictatorship delivers one. In all other cases, by computing *any* POA in polynomial time, one could answer an NP-complete decision problem in polynomial time. In Section 5 we elaborate on these easy and hard cases. For the positive results, we interpret serial dictatorship in the current problem settings. Then, we utilize a tool developed by Aziz et al. (2013) in the hardness proofs.

2. PRELIMINARIES

In this section we set the solid mathematical basis for discussing our problems. Then, we introduce a selection of NP-complete problems and prove that a specific variant of them is hard as well. We do this to prepare the ground for our hardness proofs in Sections 3-5.

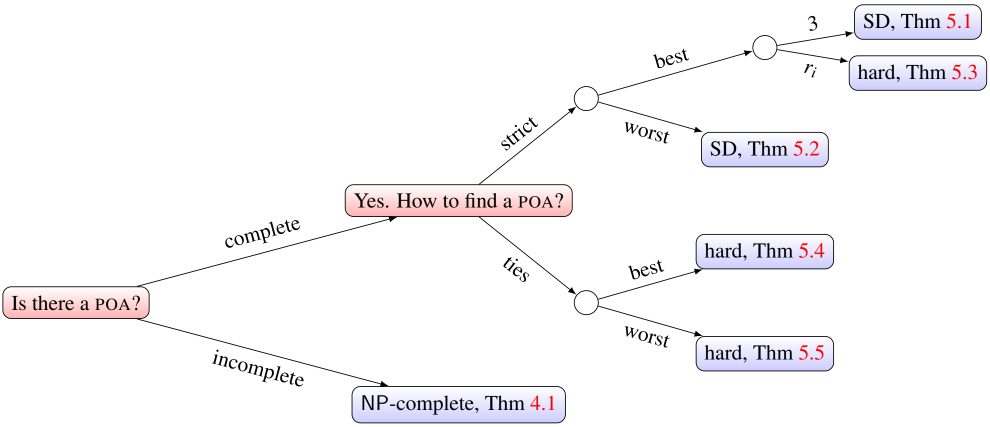

Figure 1: The complexity of finding a POA. The arrows explain the case distinctions according to points (i)–(iv) listed in the Introduction. SD stands for serial dictatorship.

2.1. Problem definition

Our input consists of a set of players P, with $|P| = n$, a multiset $r_1, r_2, \ldots, r_k$ of room capacities with $r_1 + r_2 + \ldots + r_k = n$, and strictly ordered, but not necessary complete preferences for each player $i \in P$ on other players. An assignment R is the partition of P into sets $R_1, R_2, \ldots, R_k$. This means that players have no choice to opt out once they have entered the market.

Definition 2.1. *An assignment $R = \{R_1, R_2, \ldots, R_k\}$ is called feasible if*

1. $|R_i| = r_i$ *for each* $1 \leq i \leq k$*;*
2. *each player in R_i declares every other player in the same R_i acceptable.*

Players evaluate their situation in an assignment solely based on the roommates they are grouped together with and do not care about the other rooms. For point 2 in Definition 2.1, players only need to compare outcomes in which they find all other roommates acceptable. We define two comparison principles.

When the *best* roommate counts, coalition R_i is preferred to R'_i by player i if player $j \in R$ ranked highest by i among all players in R_i is preferred by i to player $j' \in R$ ranked highest by i among all players in R'_i. Analogously, when the *worst* roommate counts, coalition R_i is preferred to R'_i by player i if player $j \in R$ ranked lowest by i among all players in R_i is preferred by i to player $j' \in R$ ranked lowest by i among all players in R'_i. Our definitions are aligned with $\mathscr{W}$-preferences in (Cechlárová & Hajduková, 2004b), and with B-hedonic and W-hedonic games in (Aziz et al., 2013). However, $\mathscr{B}$-preferences in (Cechlárová & Hajduková, 2004b) are different because there the size of the coalition serves as a tiebreaker in case the best roommate is identical in the two coalitions. We consider two such coalitions equally good.

2.2. Relevant hard problems

We first introduce two variants of a hard graph cover problem that we will later use in our hardness reductions, and then go on to consider a bin packing problem.

Problem 1. TRIANGLE COVER
Input: A simple graph $G = (V,E)$.
Question: Does there exist a partition of V into sets of cardinality 3 so that for each set, the three vertices span a 3-cycle?

Problem 2. DIRECTED TRIANGLE COVER
Input: A simple directed graph $D = (V,A)$.
Question: Does there exist a partition of V into sets of cardinality 3 so that for each set, the three vertices span a directed 3-cycle?

The TRIANGLE COVER problem asks for a set of vertex-disjoint 3-cliques (triangles) in the input graph G, so that these cover all vertices of G. This problem has been shown to be NP-complete by Garey & Johnson (1979). The directed version has not been proved to be NP-complete, so we give a simple hardness proof below. The hard problem we reduce to DIRECTED TRIANGLE COVER is 3D HYPERGRAPH MATCHING (Karp, 1972; Papadimitriou & Steiglitz, 1982).

Problem 3. 3D HYPERGRAPH MATCHING
Input: A hypergraph $H = (U \cup V \cup W, E)$, where each $e \in E$ is a triple (u, v, w) so that $u \in U$, $v \in v$, and $w \in W$.
Question: Does there exist a perfect matching in H?

Claim 1. DIRECTED TRIANGLE COVER *is an* NP*-complete problem.*

Proof. We start with the input graph of an arbitrary instance of 3D HYPERGRAPH MATCHING and transform it into an instance of DIRECTED TRIANGLE

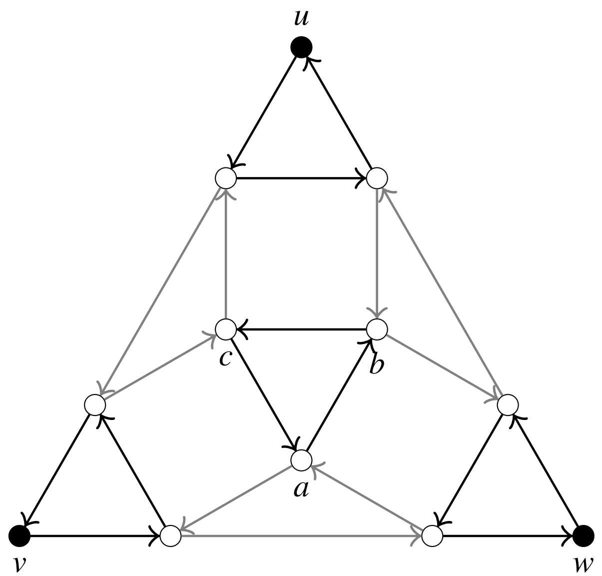

Figure 2: The gadget replacing each hyperedge $(u,v,w) \in E(H)$. The unfilled vertices are gadget vertices, and these are not connected to any vertex outside of this gadget. Triangles in this gadget are either black or gray. The gray triangles cover all of these, and leave the remaining three vertices uncovered, while the black triangles cover all vertices in the gadget.

COVER. We keep the set of vertices, and simply replace each hyperedge (u,v,w) of H by the gadget shown in Figure 2. This gadget introduces 9 new vertices per hyperedge to the set of vertices, which we will call the *gadget vertices*.

Assume first that a directed triangle cover exists in the resulting directed graph D. For each gadget, two scenarios can occur. Either vertices a,b,c form a triangle and then all of u,v, and w must be covered by triangles inside the same gadget, or triangle (a,b,c) is not in the cover, but then all of u,v, and w are covered by triangles outside the gadget. The first case corresponds to the hyperedge (u,v,w) being in our perfect matching M, while the second case tells otherwise. We still need to show that M is indeed a perfect matching. Firstly, no vertex can appear in two hyperedges in M, because it would cover the same vertex twice by directed triangles. Secondly, if a vertex is left unmatched, then it is also left uncovered by the triangle cover, which is a contradiction.

To show the opposite direction, we first assume that a perfect matching M

exists in the original instance. We transform it into directed triangles in the created directed graph D as above: the edges of M will be the gadget with (a,b,c) (the black triangles), the rest of the edges will be the gadgets without (a,b,c) (the gray triangles). We need to show that these triangles form a cover. The gadget vertices are trivially covered exactly once. The rest of the vertices are matched in M along some hyperedge, and the gadget belonging to this hyperedge covers them with a triangle inside the gadget. □

Remark 2.1. *It is an obvious observation that both* TRIANGLE COVER *and* DIRECTED TRIANGLE COVER *are hard only in graphs for which* $|V|$ *is a multiple of 3. We thus assume that G in the input of these hard problems has this property. Besides this, we can also assume that G is simple, and that each vertex is of degree at least 2, because graphs with an isolated or a degree 1 vertex are trivial no-instances. For the directed version, one can even assume that each vertex has at least one outgoing and at least one incoming edge.*

We close this section with the final element in our toolbox of hard problems. The input of UNARY BIN PACKING is a set of item sizes, and a bin size, all encoded in unary. The goal is to group all items into bins so that the total item size in each bin is exactly the bin size. This problem has been shown to be NP-complete by Garey & Johnson (1979).

Problem 4. UNARY BIN PACKING
Input: A set of item sizes $i_1, i_2, \ldots, i_n$*, and a bin size* b*, all encoded in unary.*
Question: Does there exist a partitioning of $i_1, i_2, \ldots, i_n$ *so that the sum of item sizes in each set of the partition adds up to* b*?*

Remark 2.2. *For our proofs, we assume that the smallest item size is at least 2. The hardness of this variant is easy to see. If we take an input of* UNARY BIN PACKING *and multiply all item and bin sizes by 2, then we get an equally hard problem that can be encoded in twice as many bits as the original one.*

3. VERIFICATION

In this section, we show the hardness of verification for all cases. We present two proofs: in Theorem 3.1, the worst roommate defines the base of comparison for two coalitions, while it is the best roommate who counts in Theorem 3.2. Other than this, we restrict our reduction to the least general case of the problem, having strict and complete lists, and 3-person rooms.

Theorem 3.1 (Verification, strict and complete lists, worst roommate counts, 3-person rooms)**.** *Supposing that the preferences of a player depend on the worst roommate, verifying whether a given assignment is Pareto optimal is a* coNP*-complete task even if all preferences are strict and complete, and every room is of size 3.*

Proof. For each instance G of TRIANGLE COVER we construct an instance of our verification problem, consisting of players with strict and complete preferences, and an assignment on which Pareto optimality is to be verified. We show that a triangle cover exists in the first instance if and only if the assignment has a Pareto improvement.

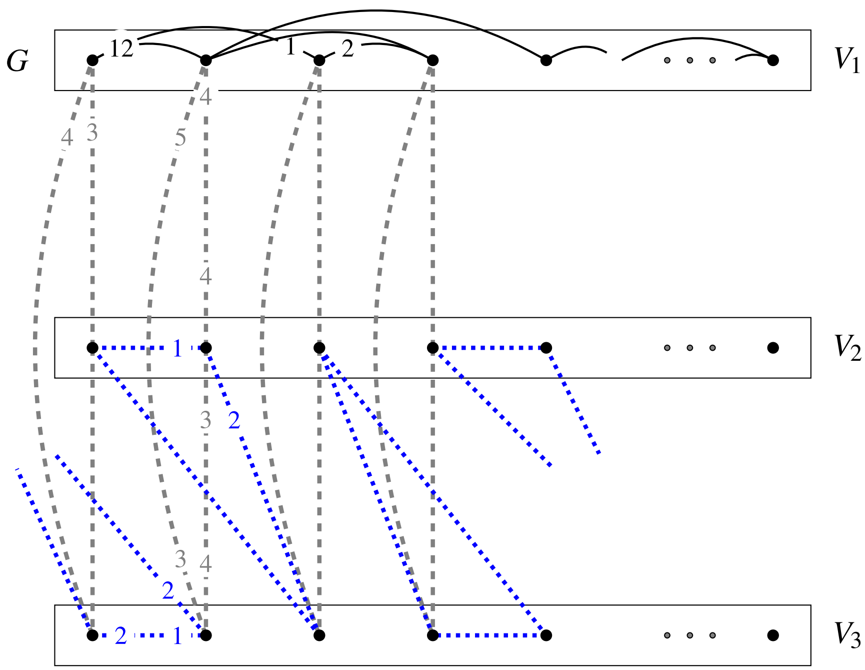

Figure 3: The assignment instance constructed to G in the proof of Theorem 3.1. The numbers on the edges mark the preferences of the players. Players in V_1 rank other players in V_1 higher than players in V_2 and V_3, as the numbers on their solid black and dashed gray edges indicate. Players in V_2 and V_3 prefer their dotted blue edges to their dashed gray edges. The dotted blue edges form a triangle cover of $V_2 \cup V_3$—they connect the leftmost two vertices in V_3 with the rightmost vertex in V_2.

First we draw graph G and also make two further copies of its vertex set V_1. We denote these copies by V_2 and V_3. Each vertex in $V_1 \cup V_2 \cup V_3$ represents a player in our POA instance. To show hardness in the most general case, we construct our POA instance with complete preference lists, which translates into a complete graph on the vertex set $V_1 \cup V_2 \cup V_3$. The edges of this graph can be partitioned into four classes.

- The original edges of G are solid and black in Figure 3. These edges are the best choices of both of their end vertices, and the order among them can be chosen arbitrarily. Recall that each vertex in V_1 has at least two of these black edges, as argued in Remark 2.1 in Section 2.

- The copied vertices V_2 and V_3 are connected by dotted blue triangles, as shown in Figure 3. Each such triangle connects three vertices that originate from three different vertices of G. Notice that these triangles can cover the entire set $V_2 \cup V_3$, because both $|V_2|$ and $|V_3|$ are multiples of 3, as noted in Remark 2.1. The rank of these edges is the highest possible, and their order among themselves does not matter.

- The three copies of the same vertex in G are connected by a dashed gray triangle in Figure 3. These edges are ranked lower than the solid black and the dotted blue edges. The order among them does not matter.

- All edges not visible in Figure 3 are ranked lower than the listed edges, in an arbitrary order.

The verification will happen with respect to the assignment R built by all dashed gray edges.

If G has a triangle cover, then these solid black triangles cover the entire set V_1. In our assignment problem, these triangles translate into coalitions of size 3. The dotted blue triangles on V_2 and V_3 complete the alternative assignment. It is easy to see that every agent is better off by switching to the solid black and the dotted blue edges, since they are always ranked higher than the dashed gray edges. Thus, R is not a POA.

Now we show the opposite. If there is a Pareto improvement to the assignment marked by the dashed gray edges, then it may use none of the invisible edges, since they are all worse than the dashed gray ones, and no player is allowed to receive a worse partner in the improved assignment. So we need to find an alternative assignment using only the first 3 types of edges. It is clear that by breaking any dashed gray coalition, at least one agent in each vertex group should be reassigned to a different coalition. Now, players in $V_2 \cup V_3$ can only choose a dotted blue edge instead of a dashed gray one. The sparsity of edges gives us that if there exists one dotted blue edge in a triangle, then not only the whole triangle is dotted blue, but we also must have all of the dotted blue triangles in the new assignment—otherwise players remain unassigned. Thus, in the new assignment, we must have all of the dotted blue triangles, and V_1's vertices are free to be grouped up among themselves. This they can only do using the solid black edges, which correspond to the original edges in G. Thus if we have a Pareto improvement of the dashed gray assignment, then we can cover V_1 with disjoint, solid black triangles. It means that G has a triangle cover. □

Theorem 3.2 (Verification, strict and complete lists, best roommate counts, 3-person rooms)**.** *Supposing that the preferences of a player depend on the best roommate, verifying whether a given assignment is Pareto optimal is a* coNP*-complete task even if all preferences are strict and complete, and every room is of size 3.*

Proof. This proof follows the lines of the previous one, but it reduces our problem to DIRECTED TRIANGLE COVER. For each instance D of DIRECTED TRIANGLE COVER we construct an instance of our verification problem, consisting of players with preferences, and an assignment on which Pareto optimality is to be verified. We show that a directed triangle cover exists in the first instance if and only if the assignment has a Pareto improvement.

First we draw the directed graph D and make two further copies of its vertex set V_1. We denote these copies by V_2 and V_3. Just as in our previous proof, the edges of this complete graph can be partitioned into four classes. Notice that the preferences differ from the ones in our previous proof.

- The original directed edges of D are solid and black in Figure 4. These edges are the best choices of their starting vertex and they are ranked below the dashed gray edges at their end vertex. The order among all outgoing and among all incoming directed edges of the same vertex can be chosen arbitrarily. Recall that each vertex in V_1 has at least one outgoing and at least one incoming edge, as stated in Remark 2.1.

- The copied vertices V_2 and V_3 are connected by dotted blue triangles, as shown in Figure 4. Each such triangle connects three vertices that

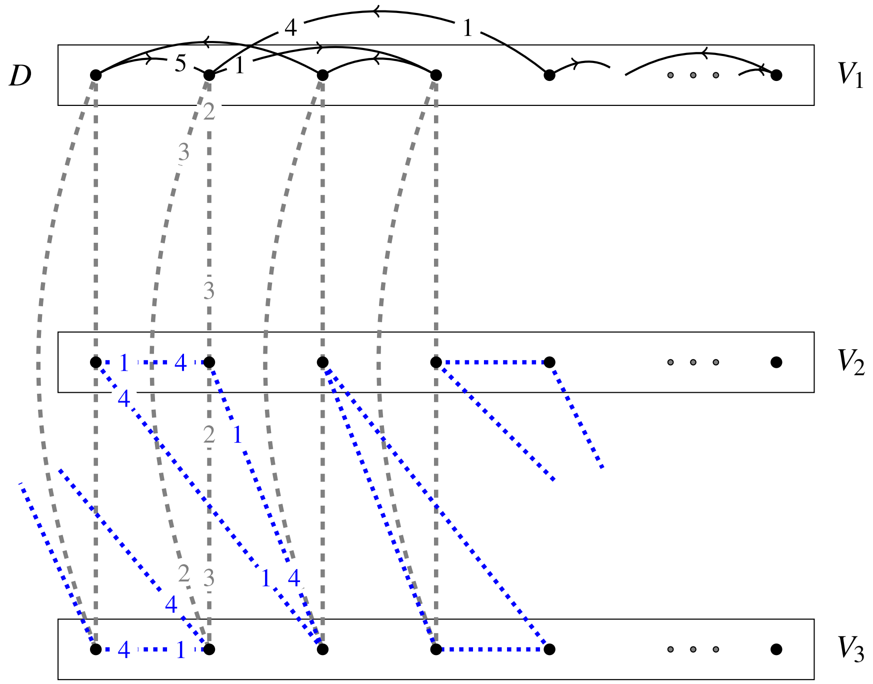

Figure 4: The assignment instance constructed to the directed graph D in the proof of Theorem 3.2. The numbers on the edges mark the preferences of the players. Players in V_1 rank their outgoing edges first, then their edges to players in V_2 and V_3, and then their incoming edges, as the numbers on their solid black and dashed gray edges indicate. Preferences in the dashed gray and dotted blue triangles are cyclic. Players in V_2 and V_3 rank their two dashed gray edges sandwiched between their two dotted blue edges.

originate from three different vertices of D. The rank of these edges is either 1 or 4, so that the triangle forms a preference cycle, i.e. each edge is ranked first by one end vertex and fourth by the other one. Edges in the middle, ranked second and third, will be the dashed gray edges.

- The three copies of the same vertex in G are connected by a dashed gray triangle in Figure 4. These edges are ranked between outgoing and incoming edges in D at vertices in V_1, and they are ranked second and third by vertices in $V_2 \cup V_3$. The order among them matters: the triangles themselves must form a preference cycle.

- All edges not visible in Figure 4 are ranked lower than the listed edges, in an arbitrary order.

Again, the verification will happen with respect to the assignment R built by all dashed gray edges.

Suppose that there exists a DIRECTED TRIANGLE COVER in D. This means that each vertex in V_1 has an outgoing edge in the triangle cover, which is a first-choice roommate in the assignment problem. So switching to the coalitions marked by the triangle partition would be a Pareto improvement for the vertices in V_1. We need to take care of the vertices in $V_2 \cup V_3$ too. The dotted blue triangles complete the alternative assignment, and due to the cyclic nature of the preferences on them, they too assign each player a first choice roommate, which was not present in the original assignment.

Now suppose that there exists a Pareto improvement to the assignment R marked by the dashed gray edges. In R, the best roommate of each player is ranked right below its outgoing black edges for vertices in V_1, and second for vertices in V_2 and V_3. In order to make at least one player, say, player i better off, the alternative assignment R' must allocate i to a more preferred neighbor j. This j must be the end vertex of the directed solid black edge (i, j), if $i \in V_1$, and i's first choice roommate (dotted blue edge) if $i \in V_2 \cup V_3$. This is only possible if we find a (non-directed) triangle partition in the constructed graph so that it contains at least one solid black or dotted blue edge, which is (i, j). We now search for a third player to complete the coalition in R'. Since j just received a bad roommate in the person of i, she must have her first or second choice in the room as well, in order to keep her satisfied. Her second choice is one of her copies j', but j' only ranks j third, and i beyond all listed players, so her situation would worsen if we put her up in a room with i and j. On the other hand, j's first choice edge (j, ℓ) is of the same color as (i, j). The position of ℓ is not worse than in the original assignment if and only if (i, j, ℓ) forms a blue or a directed black triangle. This shows that only red, blue, or directed black triangles can appear in a Pareto improved assignment. The existence of such a monochromatic triangle partition implies that there is a directed triangle cover in D. □

4. EXISTENCE

In all investigated cases, if the instance admits a feasible assignment, then it also admits a Pareto optimal one. Due to monotonicity in the rank of the worst or best roommate, a chain of Pareto improvements starting at any feasible assignment must end in a POA, which is thus guaranteed to exist if a feasible assignment exists. This is the case if lists are complete, since all assignments filling up all rooms are then feasible. This is not so for incomplete lists—by declaring some other players unacceptable, players can easily reach a situation where not even feasible assignments exist. We now show that in all cases with incomplete lists, deciding whether a feasible assignment, and thus, a POA exists, is NP-complete. Notice that since feasibility is already hard, it bears no importance whether a player judges the coalition based on the best or the worst roommate.

Theorem 4.1 (Existence, strict and incomplete lists, best or worst roommate counts, 3-person rooms)**.** *If lists are incomplete, then deciding whether a feasible assignment exists is* NP*-complete even if all preferences are strict, and every room is of size 3.*

Proof. We show hardness via a reduction from TRIANGLE COVER. Given a graph G as the input of this problem, we associate players with vertices and acceptable roommate pairs with edges. The preferences within the set of acceptable agents can be chosen arbitrarily, because they play no role in feasibility. Since feasible assignments must form coalitions of size exactly 3, each assignment in our problem corresponds to a triangle cover and vice versa. □

This theorem shows that it is computationally infeasible to find a POA if lists are incomplete, because even deciding whether there exists a POA is NP-complete. Moreover, even if we are given a feasible assignment—which guarantees the existence of a POA—it is also computationally infeasible to find a Pareto improvement, since deciding whether the given assignment itself is a POA is coNP-complete, as shown in Theorems 3.1 and 3.2. From this point on, we can thus restrict our attention to instances with complete lists.

5. FINDING A PARETO OPTIMAL ASSIGNMENT

As already mentioned in the previous section, a POA is guaranteed to exist if lists are complete. Here we distinguish all 2^3 cases based on three basic features of the problem, listed as points (ii)–(iv) in the Introduction.

5.1. Easy cases

We start by describing and analyzing the tailored variants of serial dictatorship in the cases where it delivers a POA. As usual, the core of serial dictatorship is that each dictator specifies a set of solutions that guarantee her one of her most desirable outcomes, and all later dictators must choose similarly, but within the already specified set. We show here algorithms that implement this principle under the problem settings we study.

Our algorithms are simple, and suited for the very specific problems in question, whereas the Preference Refinement Algorithm (PRA) of Aziz et al. (2013) for computing a Pareto optimal and individually rational assignment in hedonic games is generic, but also more complicated. Besides this, it invokes an oracle for solving a problem they call 'Perfect Partition'. Perfect Partition asks for an assignment that gives each player one of her best outcomes. (Notice that in this paper, we use the term perfectness in its traditional, graph-theoretical sense, where it means that each player is matched.) PRA starts off coarsening, and then refining the preferences of players according to certain rules. Then, it calls the oracle to compute a Perfect Partition with the refined preferences, which is shown to be Pareto optimal in the original instance. Here we give a much more direct interpretation of serial dictatorship, focusing only on the specific problem variant we discuss in Theorems 5.1 and 5.2. Our interpretations are then illustrated in Example 5.1.

Theorem 5.1 (Finding a POA, strict and complete lists, best roommate counts, 3-person rooms)**.** *If lists are strict and complete, all rooms are of capacity 3, and the best roommate counts, then serial dictatorship delivers a* POA.

Proof. The exact implementation of serial dictatorship works in rounds, as follows. The first dictator points at her first choice. We fix this pair, and immediately assign them their final room R_1, which will be completed by the third player later. The same happens in each round: the current dictator points at her first choice *available* roommate, we fix this pair, and assign them a room. If one player in the new pair is already in a room, the other one joins her, otherwise a new room is opened for them.

For dictator i, player j is available, if both of the following hold.

1. The number of roommates already assigned to a room together with either i or j is at most one in total, or i and j are already assigned to the same room.

2. If there is no further room to open, then j must be a player already assigned to a room.

We now show that this procedure indeed delivers a POA. Assume indirectly that an assignment $R' = R'_1, R'_2, \ldots, R'_k$ Pareto dominates the outcome $R = R_1, R_2, \ldots, R_k$ of our algorithm. Among the players who are better off in R', we choose the one who comes earliest in the order of dictators. Let this player be i. We know that j, the best roommate of i in R', is strictly better than i's best roommate in R. In i's turn in our algorithm, j thus was not an available to i. The reason for this must be one of the above two points.

If the first point was the reason, then some of the earlier dictators already reserved i and j for themselves. Since we assume that i is the earliest dictator who is better off in R' than in R, the choices of all earlier dictators must stay intact in R'. Otherwise, at least one of them receives a best roommate in R who is preferred to the best roommate of the same agent in R', which contradicts the fact that R' is a Pareto improvement. The second case is when all k rooms had already been open as i was considering to point to j, and neither of them had been assigned to a room yet. At least one of the fixed pairs must be split up in R', if i and j are assigned to the same room in it. The earlier dictator in this pair must precede $i$in the order of dictators, and thus any change in her best allocated roommate must worsen her situation, which contradicts the Pareto improvement property. □

Theorem 5.2 (Finding a POA, strict and complete lists, worst roommate counts, r_i-person rooms, including 3-person rooms)**.** *If lists are strict and complete, the rooms are of arbitrary capacity, and the worst roommate counts, then serial dictatorship delivers a* POA.

Proof. If the worst roommate counts, serial dictatorship can be interpreted as follows. In each round, the dictator moves into one of the smallest available rooms of size r_i with her best $r_i - 1$ choice roommates. The coalition is fixed and the room is removed from the set of available rooms.

To see correctness, we apply induction. Clearly, the price for improving the position of a dictator is to harm some previous dictator, because serial dictatorship gives her the fewest possible top choices still available on her list. Thus the output of the mechanism is a POA. □

Example 5.1. *Figure 5 illustrates an instance. We run serial dictatorship in the three settings in which it delivers a* POA. *We assume the order of dictators to be 1, 2, . . . , 9.*

1: 5 4 7 3 9 6 8 2	4: 3 6 7 2 9 6 8 1	7: 1 2 9 3 4 6 8 5
2: 1 4 5 9 8 6 3 7	5: 3 6 2 7 8 4 1 9	8: 6 3 7 1 9 5 4 2
3: 2 5 4 9 1 6 7 8	6: 7 2 8 5 4 9 1 3	9: 2 4 1 6 7 3 8 5

Figure 5: An instance with 9 players and strictly ordered complete lists.

- ***strict and complete lists, best roommate counts, 3-person rooms*** *(Theorem 5.1)*
  The first dictator, player 1 chooses her first choice partner player 5 and they become a fixed pair. Then, the second dictator, player 2 chooses player 1, so these three form a fixed triplet. The third dictator cannot choose her first or second choice players 2 and 5, thus she becomes a fixed pair with player 4. Now it is exactly player 4 who is next to choose, but since she is already coupled up with her first choice roommate player 3, she does not change the current assignment. Player 5 is already in a fixed room. Player 6 opens the third room together with player 7, who then adds player 9 to this room, because her first two choices are already taken. Player 8 then joins the room of player 3. The outcome is the following partition: $(1\ 2\ 5), (3\ 4\ 8), (6\ 7\ 9)$.

- ***strict and complete lists, worst roommate counts, 3-person rooms*** *(Theorem 5.2)*
  The first dictator, player 1 chooses her first and second choice partners player 5 and 4, and they occupy a room. The next dictator is player 2, whose first 3 choices are taken, thus she moves into a room with her two best available choices, players 9 and 8. The remaining 3 players are assigned to the last room. The outcome is the following partition: $(1\ 4\ 5), (2\ 8\ 9), (3\ 6\ 7)$.

- ***strict and complete lists, worst roommate counts, r_i-person rooms*** *(Theorem 5.2)*
  Let the rooms have capacity 2, 3, and 4, respectively. The first dictator, player 1 chooses her first choice partner player 5, and they occupy the smallest room. The next dictator is player 2, who cannot choose her first choice player 1, since her assignment is already fixed, thus she moves into the 3-person room with her best available choices, players 4 and 9. The remaining 4 players are assigned to the largest room. The outcome is the following partition: $(1\ 5), (2\ 4\ 9), (3\ 6\ 7\ 8)$.

5.2. Hard cases

In all remaining cases, finding a POA is computationally infeasible, even though it is guaranteed to exist. We show this in two steps, similarly to the technique used by Aziz et al. (2013). First we observe that either all POAs of an instance or none of them satisfy the property that all players receive one of their best outcomes. Then, we show that an NP-complete problem can be reduced to the decision problem of answering whether a POA exists with this property. From this it follows that by computing *any* POA in polynomial time, one could answer the NP-complete decision problem in polynomial time. Below we give three proofs for three different settings.

Theorem 5.3 (Finding a POA, strict and complete lists, best roommate counts, r_i-person rooms)**.** *If lists are strict and complete, the rooms are of arbitrary capacity, and the best roommate counts, then computing a* POA *is at least as hard as the* NP*-complete* UNARY BIN PACKING *problem.*

Proof. We construct an instance of the POA problem to each input of UNARY BIN PACKING with item size at least 2. We will show that all POAs order each player to a room with a first ranked roommate if and only if a bin packing exists.

Each item of size $i \geq 2$ (recall Remark 2.2) in UNARY BIN PACKING corresponds to i players in the POA problem. The players of one item have their unique first choice player among themselves, in a circular manner. The rest of the preference lists can be chosen arbitrarily. To guarantee that all players have a first choice roommate, we need to keep every preference cycle together. This is equivalent to keeping the items of the bin packing problem unsplit, and thus it is possible if and only if there is a perfect bin packing.

This reduction from UNARY BIN PACKING with bin size b immediately implies that the proof is valid even if all rooms are of equal size b. □

Theorem 5.4 (Finding a POA, ties, complete lists, best roommate counts, 3-person rooms)**.** *If lists are weakly ordered and complete, the rooms are of capacity 3, and the best roommate counts, then computing a* POA *is at least as hard as the* NP*-complete* DIRECTED TRIANGLE COVER *problem.*

Proof. We construct an instance of the POA problem to each input of DIRECTED TRIANGLE COVER. Let us consider a digraph D, where all vertices have at least one outgoing and at least one incoming edge, as stated in Remark 2.1. Vertices in D correspond to players in our assignment problem. If a player i has an outgoing edge towards player j in D, then i ranks j first. All other players are ranked second by i. A directed triangle cover exists in D if and only if there is an assignment where each player has at least one first choice roommate. This latter happens if and only if all POAs have this property. □

Theorem 5.5 (Finding a POA, ties, complete lists, worst roommate counts, 3-person rooms)**.** *If lists are weakly ordered and complete, the rooms are of capacity 3, and the worst roommate counts, then computing a* POA *is at least as hard as the* NP*-complete* TRIANGLE COVER *problem.*

Proof. To each input G of the TRIANGLE COVER problem, we now construct an instance of the POA problem. Remember that according to Remark 2.1, G has no isolated or degree 1 vertex. Starting with G, let us assign rank 1 to all neighbors of each player. Now we complete G by adding all missing edges and assign rank 2 on both end vertices of such an edge.

We claim that there is a POA that gives every player two of her first ranked roommates if and only if a triangle cover exists in G. If a triangle cover exists in G, then it delivers an assignment consisting of original edges only, thus it is possible to assign each player into a room with first-choice roommates only. Since each player reaches her best outcome in this assignment, it also must be Pareto optimal. To see the other direction, we assume that there is a POA that orders each player to a room with only first ranked roommates. This assignment must then consist of the edges of G exclusively, and thus it is a triangle cover in G. □

6. CONCLUSION

We have studied the complexity issues in the Pareto optimal coalition formation problem in which players have preferences over each other, and the coalitions must be of a fixed size. We have investigated a number of variants of this problem and determined the complexity of verifying Pareto optimality, deciding the existence of a POA, and finding a POA.

One natural direction of future research is to forgo the requirement on the perfectness of the assignment. In this case, due to the feasibility of the empty assignment and monotonicity, a POA trivially exists, so our question raised in Section 4 about the existence of an optimal solution does not apply. However, allowing the assignment to be imperfect leads to unnatural strategies in serial dictatorship. If the worst roommate matters, then the dictator is better off choosing her single best roommate and not letting anyone else into the room, however large it is. Besides this, one needs to clarify how to deal with the option of staying alone in a large room, which was prohibited in our setting. Depending on the total capacity of rooms, and the way of calculating the value of an imperfect coalition, one can define several natural variants of the problem. We remark that our results in Sections 3 and 5 remain valid for imperfect assignments as well, if the total capacity of rooms equals the number of players, but some agents might be left unassigned—one needs to marginally adjust serial dictatorship, while the hardness proofs carry over.

Besides this, it might be interesting to investigate analogous problems with cardinal preferences instead of ordinal ones, an outside option for players, or strategic behaviour.

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Journal of Mechanism and Institution Design
ISSN: 2399-844X (Print), 2399-8458 (Online)
DOI:10.22574/jmid.2019.11.004

A COMPARISON OF NTU VALUES IN A COOPERATIVE GAME WITH INCOMPLETE INFORMATION

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ABSTRACT

Several "value-like" solution concepts are computed and compared in a cooperative game with incomplete information and non-transferable utility. We will show that the difference between these values is caused by how payoff strategic possibilities of coalitions of the game are handled.

Keywords: Cooperative games, incomplete information, non-transferable utility.
JEL Classification Numbers: C71, D82.

1. INTRODUCTION

By introducing the concept of "virtual utility", Myerson (1984) proposed a general notion of value for cooperative games with incomplete information. The so-called M-value generalizes the Shapley non-transferable utility (NTU) value.[1] The pertinence of the M-value was tested afterwards by de Clippel (2005) in an eloquent three-player game. Later, building on Myerson's virtual utility approach, Salamanca (2019) defined an alternative value

This short paper is built upon some results from my master theses in Economics (at Toulouse School of Economics) and Applied Mathematics (at the Universidad Nacional de Colombia). I would like to thank Françoise Forges, Francisco Lozano, Michel Le Breton, Jérôme Renault, and Geoffroy de Clippel for helpful comments and discussion. This paper also benefited from the valuable comments of the editor and an anonymous referee. Any errors are, of course, my own responsibility.

[1] The Shapley NTU value is sometimes referred to as the λ-transfer value.

concept called the S-value, which generalizes the Harsanyi NTU value. Both the M-value and the S-value reflect not only the signaling costs associated with incentive compatibility, but also the fact that individuals negotiate at the interim stage (i.e., after each player has received his private information). Salamanca (2019) also showed that the M-value differs from the S-value in that the former is less sensitive to some informational externalities. In this short paper we analyze a simple example of an NTU game in which these two solution concepts differ because of the way payoff strategic possibilities of subcoalitions are handled.[2] We also study our example under the assumption of ex-ante negotiation. In that situation, the players make coalitional agreements before they acquire their private information.

2. COOPERATIVE GAMES WITH INCOMPLETE INFORMATION

The model of a cooperative game with incomplete information is as follows. Let $N = \{1,2,...,n\}$ denote the set of players. For each (non-empty) coalition $S \subseteq N$, D_S denotes the (finite) set of joint decisions for S. The sets of joint decisions are assumed to be *superadditive*, that is, for any two disjoint coalitions S and R, $D_R \times D_S \subseteq D_{R\cup S}$. For any player $i \in N$, we let T_i denote the (finite) set of possible types of player i. The interpretation is that t_i represents the private information possessed by player i. We use the notations $t_S = (t_i)_{i\in S} \in T_S = \prod_{i\in S} T_i$, $t_{-i} = t_{N\setminus i} \in T_{-i} = T_{N\setminus i}$. For simplicity we drop the subscript N in the case of the grand coalition, so we define $D := D_N$ and $T := T_N$. We assume that types are randomly chosen according to a common prior probability distribution p defined on T. The utility function of player $i \in N$ is defined to be $u_i : D\times T \to \mathbb{R}$. As in most of the literature in cooperative game theory, we assume that coalitions are *orthogonal*, namely, when coalition S chooses an action which is feasible for it, the payoffs to the members of S do not depend on the actions of the complementary coalition $N \setminus S$.

A *mechanism* for coalition $S \subseteq N$ is a pair of functions (μ_S, x_S) defined by:[3]

$$\begin{aligned} \mu_S &: T \to \Delta(D_S) & \qquad x_S &: T \to \mathbb{R}^S_- \\ t &\mapsto \mu_S(\cdot \mid t) & t &\mapsto (x^i_S(t))_{i\in S} \end{aligned}$$

[2] Our example is reminiscent of an NTU game with complete information proposed by Roth (1980).

[3] This definition is adapted from the mechanisms with sidepayments considered by Myerson (2007). For any finite set A, $\Delta(A)$ denotes the set of probability distributions over A.

Both mappings, μ_S and x_S, are measurable w.r.t. the information of the members of S. The component μ_S is a type-contingent lottery on the set of feasible decisions for S, while x_S is a vector of type-contingent utility decrements (*free disposal*). The mechanism (μ_S, x_S) $(S \neq N)$ stands as a *threat* to be carried out only if $N \setminus S$ refuses to cooperate with S. We denote by $\mathcal{F}_S$ the set of mechanisms for coalition S.

3. THE EXAMPLE

Let r be a parameter with $0 < r < 1/2$. For each value of r, we consider the following cooperative game with incomplete information: The set of players is $N = \{1,2,3\}$. Player 1 has private information about one of two possible states, $T = \{H, L\}$, which happen with prior probabilities $p(H) = 1 - p(L) = 4/5$. Feasible decisions for coalitions are $D_{\{i\}} = \{d_i\}$ $(i \in N)$, $D_{\{i,j\}} = \{[d_i, d_i], d_{ij}\}$ $(i \neq j)$, $D_N = \{[d_1, d_2, d_3], [d_{12}, d_3], [d_{13}, d_2], [d_{23}, d_1]\}$. Utility functions are given by:

(u_1, u_2, u_3)	H	L
$[d_1, d_2, d_3]$	$(0,0,0)$	$(0,0,0)$
$[d_{12}, d_3]$	$(50,50,0)$	$(40,40,0)$
$[d_{13}, d_2]$	$(100r, 0, 100(1-r))$	$(40r, 0, 40(2-r))$
$[d_{23}, d_1]$	$(0, 100r, 100(1-r))$	$(0, 40r, 40(2-r))$

Feasible decisions are understood as follows: Decision d_i denotes player i's non-cooperative option, which leaves her with her reservation utility normalized to zero. When coalition $\{i, j\}$ forms and its members agree on an outcome $d \in D_{\{i,j\}}$, player k (in the complementary coalition) is left alone with the only possibility to choose d_k. Hence, $[d_1, d_2, d_3]$ denotes the outcome in which no player cooperates, and $[d_{ij}, d_k]$ corresponds to the cooperative outcome in which players i and j form a coalition and share the proceeds of cooperation as specified above. No other outcomes are possible.

Player 3 can be considered as weak in the sense that she can only offer players 1 and 2 a payoff that is strictly lower than what they can both obtain by acting together in coalition $\{1,2\}$. Then it does appear that coalitions $\{1,3\}$ and $\{2,3\}$ are less likely to form than $\{1,2\}$. Moreover, the smaller r is, the less utility player 3 can transfer to players 1 and 2, and therefore the less likely it should be that $\{1,3\}$ or $\{2,3\}$ form.

In this game, efficient allocations can be made (Bayesian) incentive compatible. We shall thus assume that all information is public at the implementation stage, which implies that any mechanism in $\mathcal{F}_S$ can be enforced once it is agreed upon.[4] As a result, virtual utility specializes to a rescaling of actual utility and one obtains simple expressions for both the M-value and the S-value.

4. CONTRACTING AT THE INTERIM STAGE

At the interim stage each player knows her own type $t_i \in T_i$, and hence, we let $p_i(t_{-i} \mid t_i)$ denote the conditional probability of t_{-i} that player i infers given her type t_i.

For a given coalition S, we write $u_i(\mu_S, t)$ for the linear extension of the utility functions over $\mu_S(\cdot \mid t_S)$.[5] We define $u_i((\mu_S, x_S), t) := u_i(\mu_S, t) + x_S^i(t_S)$ to be player i's expected utility from (μ_S, x_S) conditional on state t. Hence, $U_i(\mu_S, x_S \mid t_i) := \sum_{t_{-i} \in T_{-i}} p_i(t_{-i} \mid t_i) u_i((\mu_S, x_S), t)$ denotes i's interim expected utility from (μ_S, x_S) given her type t_i. A mechanism $(\bar{\mu}_N, \bar{x}_N)$ is (*interim*) *efficient* for the grand coalition if there exists a non-negative vector $\lambda = (\lambda_i(t_i))_{i \in N, t_i \in T_i}$, such that $(\bar{\mu}_N, \bar{x}_N)$ maximizes the social welfare function

$$\sum_{i \in N} \sum_{t_i \in T_i} \lambda_i(t_i) U_i(\mu_N, x_N \mid t_i)$$

Thus, λ is normal to the interim Pareto frontier at the utility allocation implemented by $(\bar{\mu}_N, \bar{x}_N)$.

If we fix a vector λ of utility weights as above, then given a coalition S and a mechanism (μ_S, x_S), the *virtual utility* of player i in state t is defined as

$$v_i^\lambda((\mu_S, x_S), t) := \frac{\lambda_i(t_i)}{p_i(t_i)} u_1((\mu_S, x_S), t),$$

where $p_i(t_i)$ is i's marginal probability of her type t_i.

Consider the fictitious game in which, conditionally on every state t, virtual utilities are transferable. The worth of coalition $S \subseteq N$ in state $t \in T$, when its members agree on the mechanism (μ_S, x_S), is defined to be

$$W_S^\lambda((\mu_S, x_S), t) := \sum_{i \in S} v_i^\lambda((\mu_S, x_S), t).$$

[4] Here, the only issue is the revelation of private information at the negotiation stage.

[5] Since coalitions are orthogonal $u_i(\mu_S, t)$ is well defined.

For a given profile of threats, $\eta = ((\mu_S, x_S))_{S\subseteq N}$, $W^\lambda(\eta,t) := (W_S^\lambda((\mu_S, x_S),t))_{S\subseteq N}$ defines a TU game in state t. Let $\phi_i(W^\lambda(\eta,t))$ denote the Shapley TU value of player i in the game $W^\lambda(\eta,t)$. A mechanism $(\bar{\mu}_N, \bar{x}_N)$ for the grand coalition is (*virtually*) *equitable* if for each $t_i \in T_i$ of every $i \in N$,

$$\sum_{t_{-i}\in T_{-i}} p_i(t_{-i} \mid t_i) v_i^\lambda((\bar{\mu}_N, \bar{x}_N),t) = \sum_{t_{-i}\in T_{-i}} p_i(t_{-i} \mid t_i)\phi_i(W^\lambda(\eta,t)). \tag{1}$$

That is, an equitable mechanism $(\bar{\mu}_N, \bar{x}_N)$ gives every type of a player her (conditionally) expected Shapley TU value in the fictitious game.

A mechanism $(\bar{\mu}_N, \bar{x}_N)$ is called a *bargaining solution* if there exists a strictly positive vector[6] λ and a vector of threats $\eta = (\mu_S, x_S)_{S\subseteq N}$ such that $(\bar{\mu}_N, \bar{x}_N)$ is efficient and equitable given (λ, η). Different NTU values can be defined depending on how η is determined.

Definition 1 (M-Value). *A bargaining solution $(\bar{\mu}_N, \bar{x}_N)$ supported by λ and $\bar{\eta} = (\bar{\mu}_S, \bar{x}_S)_{S\subseteq N}$ is called an M-solution iff for every coalition $S \neq N$, $(\bar{\mu}_S, \bar{x}_S)$ solves*

$$\max_{(\mu_S, x_S)\in\mathcal{F}_S} \sum_{t\in T} p(t) W_S^\lambda((\mu_S, x_S),t). \tag{2}$$

The interim utility allocation generated by $(\bar{\mu}_N, \bar{x}_N)$ is called an M-value.

By solving (2), the M-value measures the strength of coalitions as the maximum joint gains that can be allocated inside the coalition. However, it disregards the restrictions the players face when sharing such gains.

Claim 1 (M-value). *For any given $r \in (0, 1/2)$, the unique M-value of our example is the interim utility allocation*

$$\left(U_1^H, U_1^L, U_2, U_3\right) = \left(\tfrac{100}{3}, \tfrac{80}{3}, 32, 32\right). \tag{3}$$

Proof. The interim Pareto frontier coincides with the hyperplane $\frac{4}{5}U_1^H + \frac{1}{5}U_1^L + U_2 + U_3 = 96$ on the individually rational zone. Thus, (3) is efficient. Since bargaining solutions are individually rational, an M-value can only be supported by the utility weights $(\lambda_1^H, \lambda_1^L, \lambda_2, \lambda_3) = (4/5, 1/5, 1, 1)$. Hence, virtual and real utilities coincide. After computation of threats according to (2), equations in (1) yield (3). □

[6] We focus only on non-degenerate values (i.e., those supported by strictly positive utility weights).

The M-value prescribes the same allocation regardless of the value of r. Furthermore, it treats all players symmetrically. This is due to the fact that, by computing threats according to (2), we proceed as if coalitions $\{1,3\}$ and $\{2,3\}$ could agree on an equitable distribution of the total gains, something that is not possible in the original NTU game. Thus, we may argue that threats in the M-value are not "credible".

A mechanism (μ_S, x_S) is called *egalitarian* for S w.r.t. λ and $(\mu_{S\setminus i}, x_{S\setminus i})_{i\in S}$ iff for each $t_i \in T_i$ of every $i \in S$,

$$\sum_{t_{-i}\in T_{-i}} p_i(t_{-i}\mid t_i) \sum_{j\in S\setminus i} \left[v_i^\lambda((\mu_S, x_S), t) - v_i^\lambda((\mu_{S\setminus j}, x_{S\setminus j}), t)\right]$$
$$= \sum_{t_{-i}\in T_{-i}} p_i(t_{-i}\mid t_i) \sum_{j\in S\setminus i} \left[v_j^\lambda((\mu_S, x_S), t) - v_j^\lambda((\mu_{S\setminus i}, x_{S\setminus i}), t)\right]. \quad (4)$$

That is, a mechanism (μ_S, x_S) is egalitarian if the expected average virtual contribution of the different players in S to player i equals the expected average virtual contribution of player i to the different players in S as assessed by type t_i.

For the example under consideration, a mechanism (μ_S, x_S) is egalitarian for coalition $S = \{1, j\}$ $(j = 2,3)$ if

$$v_1^\lambda((\mu_S, x_S), t) = v_j^\lambda((\mu_S, x_S), t), \quad \forall t \in T.$$

Similarly, a mechanism (μ_S, x_S) is egalitarian for coalition $S = \{2,3\}$ if

$$\sum_{t\in T} p(t) v_2^\lambda((\mu_S, x_S), t) = \sum_{t\in T} p(t) v_3^\lambda((\mu_S, x_S), t).$$

Because players 2 and 3 cannot make an agreement contingent on player 1's private information, the egalitarian equality for coalition $\{2,3\}$ holds only in expectation.

Definition 2 (S-Value). *A bargaining solution $(\bar\mu_N, \bar x_N)$ supported by λ and $\bar\eta = (\bar\mu_S, \bar x_S)_{S\subseteq N}$ is called an S-solution iff for every coalition $S \neq N$, $(\bar\mu_S, \bar x_S)$ solves*

$$\max_{(\mu_S, x_S)\in\mathcal{F}_S} \sum_{t\in T} p(t) W_S^\lambda((\mu_S, x_S), t) \quad (5)$$
$$\text{s.t. (4) w.r.t. } \lambda \text{ and } (\bar\mu_{S\setminus i}, \bar x_{S\setminus i})_{i\in S}$$

The interim utility allocation generated by $(\bar\mu_N, \bar x_N)$ is called an S-value.

By imposing the egalitarian constraints, the S-value takes into account the equity restrictions that coalitions face when sharing the proceeds of cooperation.

Claim 2 (S-value). *For a given $r \in (0,1/2)$, the unique S-value of our example is the interim utility allocation*

$$\left(U_1^H, U_1^L, U_2, U_3\right) = \left(50 - \tfrac{100}{3} r \left(\tfrac{88-88r}{96-88r}\right), 40 - \tfrac{80}{3} r \left(\tfrac{88-44r}{96-88r}\right), 48 - \tfrac{88}{3} r, \tfrac{176}{3} r\right). \quad (6)$$

Proof. The same reasoning as in the proof of Claim 1. □

The S-value gives less to player 3 compared to the M-value. This is due to the fact that two-person coalitions with player 3 cannot fully distribute the total gains from cooperation in an equitable way. This lack of transferability increases as long as r decreases to 0, which explains why the S-value converges to the allocation $(50,40,48,0)$ as r vanishes. It seems that the S-value reflects the power structure of this game better than the M-value, in particular for a small r.

5. CONTRACTING AT THE EX-ANTE STAGE

When contracting takes place at the ex-ante stage, players face a cooperative game under incomplete information but with symmetric uncertainty. Then we may apply both the Shapley NTU value and the Harsanyi NTU value to the associated characteristic function of the game.

We let $U_i(\mu_S, x_S) := \sum_{t \in T} p(t) u_i((\mu_S, x_S), t)$ denote i's ex-ante expected utility from (μ_S, x_S). The set of feasible payoff allocations for coalition $S \subseteq N$ is given by $V(S) = \{(U_i(\mu_S, x_S))_{i \in S} \mid (\mu_S, x_S) \in \mathcal{F}_S\}$. Then the ex-ante characteristic function of our example is:

$$\begin{aligned}
V_r(\{i\}) &= \{u_i \mid u_i \leq 0\}, \quad \forall i \in N, \\
V_r(\{1,2\}) &= \{(u_1, u_2) \mid u_1 \leq 48,\ u_2 \leq 48\}, \\
V_r(\{i,3\}) &= \{(u_i, u_3) \mid u_i \leq 88r,\ u_3 \leq 96 - 88r\},\ (i = 1,2), \\
V_r(N) &= comp\,(\{u_{12}, u_{13}, u_{23}\}),
\end{aligned}$$

where $u_{13} := (88r, 0, 96-88r)$, $u_{23} := (0, 88r, 96-88r)$, $u_{12} := (48,48,0)$ and, for any finite set A, $comp\,(A)$ denotes the closed comprehensive hull of A.

Definition 3 (Shapley NTU value). *A payoff configuration $u = (u_S)_{S \subseteq N}$, with $u_S \in V(S)$, is a Shapley NTU solution of a game (V,N) if there exists a vector of strictly positive utility weights $\lambda = (\lambda_i)_{i \in N}$ such that:*

$$\sum_{i \in S} \lambda_i u_S^i \geq \sum_{i \in S} \lambda_i v^i, \quad \forall v \in V(S),$$

$$\lambda_i u_N^i = \phi_i(w^\lambda), \quad \forall i \in N,$$

where w^λ is the TU game defined by $w^\lambda(S) = \sum_{i \in S} \lambda_i u_S^i$, for every $S \subseteq N$. The resulting allocation u_N is called a Shapley NTU value of (V,N).

Claim 3 (Ex-ante Shapley NTU value). *For every $r \in (0,1/2)$, the unique Shapley NTU value of the game (V_r,N) is the (ex-ante) utility allocation*

$$(U_1,U_2,U_3) = (32,32,32). \tag{7}$$

Like the M-value, the Shapley NTU value is independent of r. Moreover, it treats all players symmetrically and ignores the fact that coalitions $\{1,3\}$ and $\{2,3\}$ cannot agree on an equitable distribution of the gains.

Definition 4 (Harsanyi NTU value). *A payoff configuration $u = (u_S)_{S \subseteq N}$, with $u_S \in V(S)$, is a Harsanyi NTU solution of a game (V,N) if there exists a vector of strictly positive utility weights $\lambda = (\lambda_i)_{i \in N}$ such that:*

$$u_S \in \partial V(S), \quad \forall S \subseteq N,$$

$$\sum_{i \in N} \lambda_i u_N^i \geq \sum_{i \in N} \lambda_i v^i, \quad \forall v \in V(N),$$

$$\lambda_i \left(u_S^i - u_{S \setminus j}^i\right) = \lambda_j \left(u_S^j - u_{S \setminus i}^j\right), \quad \forall i,j \in S, \forall S \subseteq N,$$

where $\partial V(S)$ denotes the (Pareto efficient) boundary of $V(S)$. The resulting allocation u_N is called a Harsanyi NTU value of (V,N).

Claim 4 (Ex-ante Harsanyi NTU value). *For a given $r \in (0,1/2)$, the unique Harsanyi NTU value of the game (V_r,N) is the (ex-ante) utility allocation*

$$(U_1,U_2,U_3) = \left(1 - \frac{22r}{36-33r}\right) u_{12} + \frac{11r}{36-33r} u_{13} + \frac{11r}{36-33r} u_{23}. \tag{8}$$

For every r, the weight of the outcome u_{12} of coalition $\{1,2\}$ is the largest. Furthermore, it increases to 1 as r decreases to 0; thus the probability of player 3 getting into a coalition converges to 0. Therefore, the Harsanyi NTU value prescribes an outcome that better captures the lack of transferable utility in this game.

6. CONCLUDING REMARKS

We have analyzed a simple example of an NTU game with incomplete information where the M-value and the S-value yield different outcomes. The computations leading to these outcomes show that the difference between them derives from the way each value measures the strength of intermediate coalitions: while the M-value only reflects the total gains from cooperation, the S-value recognizes, in addition, the restrictions that players face when sharing such proceeds. Similar conclusions are also obtained for the Shapley and Harsanyi NTU values when contracting takes place at the ex-ante stage. We hope that the analysis here will shed further light on the interpretation of the values of cooperative games with incomplete information.

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Journal of Mechanism and Institution Design
ISSN: 2399-844X (Print), 2399-8458 (Online)
DOI:10.22574/jmid.2019.11.005

LOSS AVERSION IN FINANCIAL MARKETS

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ABSTRACT

Experimental evidence suggests that people are more sensitive to losses than gains by a factor of about two. Researchers have drawn implications from loss aversion to understand various aspects of individual decisions and asset prices in financial markets. At the current stage, some ancillary assumptions have been made in deriving these implications. Loss aversion affects financial markets through affecting the risk attitudes of market participants. Taken as a whole, loss aversion is a useful ingredient in helping us understand financial markets.

Keywords: Loss aversion, risk attitude, anomalies.
JEL Classification Numbers: G11, G12, D50.

> "A central conclusion of the study of risky choice has been that such choices are best explained by assuming that the significant carriers of utility are not states of wealth or welfare, but changes relative to a neutral reference point. Another central result is that changes that make things worse (losses) loom larger than improvements or gains." (Kahneman et al., 1991, p.199)

1. INTRODUCTION

In financial economics, most theories assume that investors evaluate risks according to the expected-utility framework.[1] However, researchers find

[1] The expected-utility framework was developed by Von Neumann & Morgenstern (1947) and Savage (1954). Under this framework, an agent derives utility from final wealth levels w at different states according to a concave utility function $u(w)$. For a random wealth W, its value according to the expected utility is $E[u(W)]$, where $E[\cdot]$ is the expectation operator under the distribution of W.

that the behavior of individuals and aggregate outcomes in financial markets are sometimes inconsistent with predictions derived from expected-utility maximizers. These observations have motivated the study of various alternative decision theories. One particularly interesting alternative theory is prospect theory, which was proposed in 1979 based on extensive experimental evidence by two psychologists, Amos Tversky and Daniel Kahneman. It is widely believed that this theory offers a sensible way of understanding how people think about risk.

One of the most salient features of prospect theory is loss aversion,[2] which was introduced to capture the experimental evidence that people tend to reject lotteries such as a 50:50 bet to win \$110 or lose \$100. Kahneman & Tversky (1979) posited that people get both pleasure and pain directly from gains and losses and that they are more sensitive to losses than gains. Formally, people evaluate gains and losses using the following value function:

$$v(x) = \begin{cases} x, & if\ x \geq 0, \\ \lambda x, & if\ x < 0, \end{cases}$$

where $x > 0$ denotes gains and $x < 0$ denotes losses. Parameter $\lambda > 1$ controls the degree of loss aversion; its empirical value is close to 2, and we will use this value in this article. When facing a lottery, people first apply this value function to each possible gain and loss, then multiply the respective probability and sum up to arrive at a final value to evaluate how attractive the lottery is. That is, for any lottery X, its value according to loss-aversion utility is $E[v(X)]$. For example, for the 50:50 bet to win \$110 or lose \$100, the value is:

$$\text{value of taking the bet} = 110 * 0.5 - 2 * 100 * 0.5 = -45,$$

and thus a loss-averse person would turn down this bet.[3]

Over the past few decades, researchers have made significant progress in applying loss aversion to understanding both individual behaviors—such as trading assets in financial markets and making real investments within a

[2] Many other theories, such as disappointment aversion or regret theory, also share similar features to loss aversion.

[3] Alternatively, consider an expected-utility maximizer. Suppose that she has an initial wealth of \$75,000 and has a smooth, concave expected-utility function $u(w) = \ln(w)$. If she turns down the bet, the value is ln(75,000) = 11.2252. If she takes the bet, the value is ln(75,000 + 110)*0.5 + ln(75,000 – 100)*0.5 = 11.2253. As a result, this expected-utility maximizer would take the bet.

company—as well as aggregate market outcomes such as asset prices. For financial investors, loss aversion helps to explain their reluctance to allocate any money to the stock market (the "nonparticipation puzzle"), their refusal to hold a well-diversified portfolio if they decide to participate in the stock market at all (the "under-diversification puzzle"), and their tendency to sell better performing stocks than poorer performing ones (the "disposition effect"). For corporate company managers, loss aversion helps to explain their reluctance to shut down projects that are doing poorly, as well as other real decisions such as mergers and acquisitions and dividends payouts. For asset prices, loss aversion helps to explain the high average return on the stock market (the "equity premium puzzle"), the high volatility of stock returns (the "excess volatility puzzle"), the volatility clustering feature of stock returns (the "GARCH effect"), as well as the fact that stock returns can be forecast by various factors in both a time series and cross section.

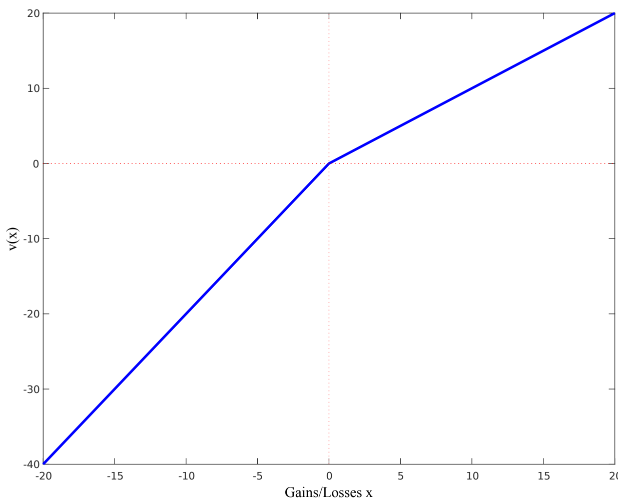

Figure 1: The value function of loss aversion.

Essentially, loss aversion affects the financial market through influencing people's attitude toward risks. To clarify this point, I use Figure 1 to plot the value function $v(x)$ for $\lambda = 2$. Loss aversion affects people's risk attitude in three ways. First, the value function in Figure 1 is *globally* concave, meaning that a loss-averse investor is averse to risks (in economics terms, she behaves as if she is risk averse).[4] This implies that the investor is reluctant to accept a bargain with an uncertain payoff rather than another bargain that has more certainty but also a possibly lower expected payoff. When it comes to the financial market, a loss-averse investor is more inclined to put her money into T-bills with a low but relatively guaranteed return rather than into a stock that has high expected returns, but one that also involves a high chance of losing value. As a result, this global risk aversion induced by loss aversion helps to explain the nonparticipation puzzle, and it can generate a high equity premium in an equilibrium model.

Second, the value function has a "kink" at the origin. This feature is known as "first-order risk aversion" in economic theory. Informally, it means that the investor is *locally* risk averse, while the traditional expected utility exhibits local risk neutrality. The kink often implies that the investor's optimal decision is sluggish in response to exogenous changes in environments. In terms of financial applications, this feature implies that companies tend to cluster their dividend payouts at a certain level, that potential acquirers tend to cluster their offer prices around a historical high stock price of a targeted firm, and that loss-averse traders choose a zero holding of the risky asset when they trade against non-loss-averse traders.

Third, in a dynamic setting, loss aversion can cause a time-varying risk attitude. Researchers have identified this changing risk aversion generally through two approaches—by borrowing psychological evidence which suggests that sensitivity to loss is dependent on prior gains and losses, or by pinpointing the mismatching of the decision intervals and the evaluation intervals (for example, an investor trades stocks every month over the year, but she only derives loss-aversion utility from the cumulative gains and losses at the year end). A time-varying risk attitude can generate excess volatility and predictability of stock returns through affecting how much compensation a loss-averse investor would require for holding a risky asset.

In the rest of this article, I first present the challenges that researchers face in applying loss aversion in financial markets, then discuss the implications of

[4] Informally, "risk aversion" refers to a general preference for safety over uncertainty in potential gains or losses. Loss aversion emphasizes a strong distaste for losing money. In economics, risk aversion is captured by the curvature of the expected utility value function $u(w)$, where w is the agent's *overall wealth level*. The u function can be related to the value function v of loss aversion. For instance, if the reference point is given by some initial wealth level w_0 so that the gain or loss x is simply $w - w_0$, then one can link v and u as follows: $v(x) = v(w - w_0) \equiv u(w)$.

loss aversion for individual decisions and asset prices, and finally make some concluding remarks. I want to emphasize that this article is not intended to conduct a complete survey on loss aversion[5] but instead to illustrate how loss aversion can be applied to help us understand the financial market. I hope that the material in this article can inspire more fruitful research in the area of loss aversion in financial markets and guide market design in scenarios in which loss aversion is present.

2. THE DIFFICULTY OF APPLYING LOSS AVERSION IN FINANCE

When researchers apply loss aversion to determine implications for financial markets, they need to make many auxiliary assumptions as it is often unclear how to exactly define gains or losses. There are four related issues here. The first issue is determining whether people engage in "mental accounting" or "narrow framing" when they evaluate their financial investments in multiple assets. For example, suppose that you own a portfolio of two stocks, IBM and Apple. Over the course of the year, your IBM holdings go up in value by \$1,000 while your Apple holdings go down by \$1,000. You can frame your investment performance in two ways: by engaging in broad framing, you can look at your overall portfolio performance and thus register \$0 gain/loss for this broad account; alternatively, you can engage in narrow framing and look at the performance of each stock separately and, in this scenario, feel good about your investment in IBM but feel really bad about your investment in Apple.

The second issue concerns reference points. Again, think about the above example. Say that the framing issue is settled: you accept narrow framing and look at each stock in isolation. But it is still unclear how exactly to measure gains and losses: Does a "gain" mean that the value of the investment exceeds its purchase price, or exceeds its purchase price adjusted by some rate, or exceeds some other targets such as historical high prices or expected future prices? Say the reference point is the purchase price adjusted by the average return on T-bills (perhaps because if the money was put into T-bills, the investment value would have increased on this rate). By a similar reason, an adjustment can be made based on any rate on similar alternatives or the

[5] For excellent surveys on prospect theory, or more broadly, behavioral finance, see, for example, Barberis (2013), Barberis (2018), Barberis & Thaler (2003), Kahneman (2003), and Hirshleifer (2015).

expected return on the stock under consideration. This reference point issue can become more complicated in a dynamic setting because it often involves the updates of reference points. For instance, suppose that you have bought in Apple at a price of $500 per share and the stock has been below that value for quite a while. Would you still keep $500 as a reference point or would you adjust it downward since the price has continued to decline for a while?

The third issue is about how frequently investors evaluate their gains or losses. Should gains/losses be checked annually or monthly? Or is there no fixed frequency and an evaluation should be conducted once a trade is completed? Relatedly, trading frequency can be quite different from evaluation frequency, and so temporal framing exists here as well. Consider the following example. Suppose that you purchased a stock at $1,000 at the beginning of the year. In June, its value goes down to $500, but in December, its value goes up to $1,500. Would you count this investment as a total gain of $500 (you will if you derive utility from annual gains and losses), or count it as a loss of $500 and a gain of $1,000 (you will if you think about gains and losses on a semi-annual frequency)? This issue is similar to the first narrow-framing issue, but in that scenario the issue is about evaluating across different assets while here it is about evaluating over different time periods for a given stock (in this sense, this issue can be thought of as an issue of temporal narrow framing).

Finally, the last issue is about whether trading profits have to be realized in order to be counted as a gain or loss. Would a paper loss generate a negative utility? Some people may feel bad even if they only watch their financial wealth fluctuate while for others an unrealized loss is not that painful as long as it is just on paper ("it's only a paper loss—it'll come back").

In all finance applications of loss aversion, researchers have to make assumptions about the above four issues. Most researchers have adopted the assumption of narrow framing or mental accounting and have used purchase prices or a risk-free rate as reference points. For evaluation frequency, some researchers argue that annual frequency seems to be most natural, perhaps because the performance of most asset classes is often reported in annual terms. Most of the studies have also assumed that people care about paper gains or losses, which is particularly true for asset-pricing models, because in these models, there is typically no trading and thus no gains or losses are realized.[6]

[6] Barberis & Xiong (2012) and Ingersoll & Jin (2013) have studied the implications of "realization utility" in which people derive utility only from realized gains or losses.

3. IMPLICATIONS FOR INDIVIDUAL BEHAVIORS

3.1. Nonparticipation in the Stock Market

For much of the twentieth century, most households did not invest any money in stocks, even though the stock market has offered a high mean rate of return. For instance, in 1984, only 28% of U.S. households held any stock, and only 12% held more than \$10,000 in stock. Today, the fraction of households that own stock is close to 50%. This behavior is puzzling because standard finance theory predicts that people should invest at least some money in the stock market, which offers an actuarially favorable gamble. Researchers have argued that loss aversion is useful for explaining the nonparticipation puzzle: loss aversion means that investors are more sensitive to losses than to gains and since stock returns are volatile, holding stocks would make investors often face losses and thus they are reluctant to invest in the stock market.

An example is helpful. In the U.S., historically, the annual stock market return has a mean of 6% and a volatility of 20%. To match this data, let us suppose that if a current investor puts \$10,000 in a stock, the next year she may gain \$2,600 or lose \$1,400 with equal probability. To compute the value that a loss-averse investor would place on this investment, let us assume that she uses the purchase price as the reference point and computes loss-aversion utility based on the trading profit over the year. So this investor would simply multiply each outcome by its probability and keep in mind that there is double sensitivity to losses (i.e., $\lambda = 2$), yielding:

$$\text{value of participating in stock markets} = 2,600*0.5-2*1,400*0.5 = -100,$$

which is smaller than 0, or the value that she would place if there was no investment in the stock market. This explains the nonparticipation puzzle.

Barberis et al. (2006) caution that the above argument has implicitly incorporated the important assumption of "narrow framing;" that is, when people think about whether to invest in the stock market, they may be thinking about it in isolation rather than in combination with other risks that they are already facing such as fluctuations in labor income and house prices. If we extend this narrow-framing assumption to an asset class level or to an individual stock level, then by the same logic, loss aversion is also useful for understanding other related puzzles: the home bias puzzle, the refusal of many households to invest in international equities; and the under-diversification puzzle, which is the tendency of some stockholders to hold only a small number of stocks, rather than many stocks for the benefits of diversification.

3.2. The Disposition Effect

Another often-studied financial phenomenon is the "disposition effect," which concerns how people trade assets over time. The disposition effect posits that both individual investors and mutual fund managers have a greater tendency to sell assets that have risen in value since the purchase than those that have fallen. This effect has been repeatedly observed both in experiments and in real markets in which people trade stocks, houses, or stock options. It is puzzling because most obvious rational explanations, such as portfolio rebalancing or information story, cannot entirely account for the effect. For example, stock markets exhibit "momentum"—i.e., stocks that have done well over the past six months continue to outperform other stocks, on average, while stocks that have done poorly over the past six months continue to lag. As such, the rational thing to do is to retain stocks that have recently risen in value and sell those that have recently fallen in value, which is the opposite of the disposition effect.

The literature has pointed to loss aversion as a potential ingredient underlying the disposition effect.[7] For example, in explaining disposition behavior in the Boston condominium market, Genesove & Mayer (2001, p.1235) wrote: "When house prices fall after a boom, as in Boston, many units have a market value below what the current owner paid for them. Owners who are *averse to losses* will have an incentive to attenuate that loss by deciding upon a reservation price that exceeds the level they would set in the absence of a loss, and so set a higher asking price, spend a longer time on the market, and receive a higher transaction price upon a sale." Recent theoretical research, however, suggests that the link between loss aversion and the disposition effect is more nuanced.

Li & Yang (2013a) find that loss aversion can either drive a disposition effect or a reversed disposition effect (that people are more inclined to sell losers than winners), depending on the skewness of the stock return process.[8]

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7 Another component of prospect theory is diminishing sensitivity, that is, people are risk averse over gains and risk seeking over losses, which has also been commonly cited to explain the disposition effect.

8 Skewness is computed as the third standardized moment of stock returns. It is a measure of the asymmetry of the return distribution about the mean. When the return distribution is symmetric, its skewness is zero. Negative skewness suggests that the return distribution is

In their setting, a loss-averse investor buys shares of a stock at the beginning of the year. Over the course of the year, she trades the stock and at the end of the year receives loss-aversion utility based on her trading profit. In this setting, the underlying link between loss aversion and the selling behavior is essentially a time-varying risk attitude story. Loss aversion means that the value function has a kink at the origin and the investor is afraid of holding stocks if she is close to the kink. In Fig. 1, the value function is locally concave at the origin (the investor is locally risk averse when her current gain or loss is close to zero), while it is locally linear at a deep gain or loss (the investor is locally risk neutral when she has already accumulated many gains or losses). Whether loss aversion drives a disposition effect depends on whether good news or bad news moves the investor closer to the kink. Now suppose that the stock return process is negatively skewed. The odds of bad news are then very small but once the news occurs, its size is very large, dragging the investor down to a deep loss at a position very far away from the kink. In contrast, good news occurs more frequently but of a small order, leaving the investor close to the kink when facing gains. As a result, with bad news, the investor will be less afraid of risk and more likely to hold the stock, thus generating the disposition effect. In the case of a positively skewed stock return process, the opposite happens, leading to a reversed disposition effect.[9] Again, let us use an example. Suppose that you have invested \$10,000 in a stock in January and you will evaluate this investment based on the cumulative gains or losses at the end of December in a way as described by loss aversion. Let us also suppose that over any six-month period, the stock return can be either 11% with a probability of 0.75, or -23% with a probability of 0.25. This return distribution is left-tailed with a skewness of -1.15. Now let us fast forward six months to June and suppose that the stock investment has gone up to \$11,100 (=10,000*(1+11%)). What will you do now? If you sell the stock, you can

left-tailed, that is, for most of the time, stock returns are positive and relatively small, but with small probabilities, there can be quite large price drops. Positive skewness suggests the opposite.

[9] Li & Yang (2013a) specify that the reference point is the stock's purchase price or equivalently the investor's initial wealth. Meng & Weng (2018) show that if the reference point is based on expected wealth rather than initial wealth, prospect theory can generate a disposition effect even for non-skewed stock return processes. Pagel (2016) suggests that expectation-based reference points are also relevant for understanding asset price behaviors.

lock in a gain of $1,100 and its value under loss-aversion utility is

$$\text{value of selling} = 1,100.$$

In contrast, if you decide to keep the stock and if you're lucky, the stock will continue to go up and at the end of December your investment will become $12,321 (=11,100*(1+11%)). This returns a total gain of $2,321, with a probability of 0.75. If you are unlucky and the stock drops in December, your investment will drop to $8,547 (=11,100*(1-23%)) with a probability of 0.25, generating a total loss of $1,453. As a result, the value is:

$$\text{value of holding} = 2,321 * 0.75 - 2 * 1,453 * 0.25 = 1014.30,$$

which is smaller than the value 1,100 of selling. So you may want to sell the stock in June after it has gone up.

Now suppose that in June the stock has gone down to $7,700 (=10,000*(1-23%)). If you sell the stock, you will incur a loss of $2,300, which implies

$$\text{value of selling} = -2 * 2,300 = -4,600.$$

By contrast, if you hold the stock until December, then the stock can go up to $8,547 (=7,700*(1+11%)) with a probability of 0.75, leaving a relatively small overall loss of $1,453, or it can continue to drop to $5,929 (=7,700*(1-23%)) with a probability of 0.25, leading to a large total loss of $4,071. Thus, the loss-aversion utility of keeping the stock is

$$\text{value of holding} = -2 * 0.75 * 1,453 - 2 * 0.25 * 4,071 = -4,215,$$

which is larger than the value -4,600 if you sold the stock. Thus, you may want to keep the stock after the stock has declined in June. So, facing bad news in June, you hold on to the stock, while facing good news in June, you close the position, which is the disposition effect. Note that underlying this effect is that bad news moves you further away from the kink with a relatively large loss of $2,300 than good news with a relatively small gain of $1,100.

3.3. Corporate Decisions

Researchers find that loss aversion is also useful in understanding corporate decisions that affect real resource allocations. For example, similar to the disposition effect in the financial market, managers are reluctant to divest or shut down projects that are doing poorly. So, as loss aversion explains the disposition effect, it can also justify a reluctance to terminate bad real investments.

Baker & Wurgler (2013) argue that loss aversion is a useful ingredient for understanding dividend policies. They construct a dynamic model in which a benevolent firm manager decides how many dividends to pay out to her investors. Investors are loss averse in the sense that the negative effect of a $1 drop in dividends below the current reference point is greater than the positive effect of a $1 increase in dividends. In their model, the reference point in the current period is the dividend level set in the previous period. So although increasing dividends today have the benefit of avoiding falling short of today's reference, they also have the cost of raising tomorrow's reference point and the likelihood of generating future losses. The optimal dividend policy is determined by this tradeoff. Because of the loss-aversion kink feature, Baker & Wurgler (2013) find that the optimal dividend payout often clusters at a certain level and adjusts only when earnings are sufficiently large. This prediction is largely consistent with existing empirical literature.

Baker et al. (2012) suggest that loss aversion also helps to understand various aspects of merger and acquisition activity. They argue that historical peak prices, such as the 13-week high or the 52-week high, can serve as the natural reference point for an offer price because these peak prices are routinely reported and discussed in the financial press and thus salient to all related parties. They find that in their sample, offer prices cluster around these peak prices. In their interpretation, one important element is that loss aversion affects both the psychology of the target's management and shareholders and that of the bidder's management. For instance, from the target's perspective, the reluctance to realize losses relative to a reference point (i.e., the disposition effect in the trading behavior reviewed in the previous section) implies that targets are more likely to approve mergers in which the offer price approaches or exceeds a recent peak price (as a reference price). Finally, Baker et al. (2012) also find that their anchoring story can explain bidder announcement effects, deal successes, and merger waves.

4. IMPLICATIONS FOR ASSET PRICES

4.1. The Equity Premium Puzzle

The equity premium puzzle has to do with the fact that the average return on the aggregate stock market has historically been much higher than the average return on Treasury bills over the past century. The equity premium can range from 4% to 8% on an annual basis, depending on the source datasets. Of course, since the stock market is riskier than T-bills, the average stock return should be higher than the average return on T-bills. The problem is that traditional models can only generate an annual equity premium smaller than 0.5%. Thus, the equity premium puzzle is a quantity puzzle: Why has the historical equity premium been so consistently high?

One of the most robust predictions of loss aversion is a sizeable equity premium in an equilibrium model. The intuition is simple. Loss aversion means that investors are more sensitive to losses than to gains. Since stocks often perform poorly and thus investors often face losses, a large premium is required to convince them to hold stocks. Since the equity premium puzzle is about magnitude, let us use the following example—in the spirit of Benartzi & Thaler (1995) calibration exercise—to illustrate how loss aversion can generate an equity premium as high as its historical value.

For simplicity, let us suppose that the T-bill rate is zero so that the equity premium is the average stock return. Suppose that the stock market can go up or down so that the annual stock return can take two values, $r_u > 0$ and $r_d < 0$, with equal probability. Under this distribution, the average and volatility of stock returns are

$$\text{ret.avg.} = 0.5 * (r_u + r_d) \text{ and } \text{ret.vol.} = 0.5 * (r_u - r_d).$$

We fix the return volatility at its historical value 20% (i.e., $0.5 * (r_u - r_d) = 20\%$), and check whether loss aversion can deliver a high average return. Suppose that investors are loss averse in financial markets; they like gains and dislike losses and they are doubly sensitive to losses than to gains. Since these investors hold both T-bills and stocks, they must be indifferent to holding these two assets. Holding the T-bill delivers no gains or losses and thus the value is zero, which in turn means that the value of holding the stock has to be zero as well. Since investing \$1 in the stock will deliver a gain of $r_u > 0$, or a loss of $r_d < 0$, with equal probability, its value is

$$\text{value of holding the stock} = 0.5 * r_u + 2 * 0.5 r_d = 0.$$

Combining the above equation with $00.5*(r_u - r_d) = 20\%$, we can compute $r_u = 27\%$ and $r_d = -13\%$, which implies an annual equity premium of $0.5*(r_u + r_d) = 7\%$!

4.2. Return Volatility

Stock return volatility has two salient features. First, its level is surprisingly high, relative to the volatility of dividends (which are a proxy for fundamentals). For instance, the U.S. stock market has a return volatility of 20% on an annual basis while the volatility of dividend growth is around 6% for the post-war period. This fact is termed the "excess volatility puzzle" in the literature. Second, return volatility is highly persistent, or in statistical terms, return-volatility autocorrelation is very high, which is known as a "GARCH effect." Researchers have found that loss aversion can shed light on both features of return volatility.

The reason that loss aversion helps to explain the excess volatility puzzle is that loss aversion can generate a time-varying risk aversion in a way such that people are more risk averse after bad news than after good news. The intuition is straightforward—after bad (good) news, the stock market goes down (up); people thus become more (less) risk averse and more (less) anxious holding the stock, pushing the stock price down (up) even further, which amplifies the volatility of the stock market to a greater extent than can be justified by the volatility of fundamentals alone.

Again, let us use an example to illustrate this point. Suppose that a stock was worth \$33.67 per share last year and that it paid a dividend of \$1.00. Also suppose that each year it is equally likely that the stock's dividend can increase either by 7% or decrease by 5% so that the average dividend growth rate is $\mu_D = 0.5*(7\% - 5\%) = 1\%$ and the volatility of dividend growth rate is $\sigma_D = 0.5*(7\% + 5\%) = 6\%$. In a traditional model with constant risk aversion, the discount rate would be constant, so let us assume that it is 4%. Let us also assume that the stock is priced according to a standard textbook Gordon formula as follows:

$$\text{price of the stock}=\frac{(1\text{+average dividend growth rate }\mu_D)\times\text{current dividend}}{\text{discount rate - average dividend growth rate }\mu_D}.$$

Now, suppose that the dividend news is good this year so that the stock's underlying firm issues a dividend of \$1.07. According to the Gordon formula, the stock price rises to $\frac{(1+1\%)\$1.07}{4\%-1\%} = \$36.02$, where the cash flow is discounted at the rate of 4%. So, given the good dividend news this year, the stock return is

$$r_u = \frac{\$36.02 + \$1.07}{\$33.67} - 1 \approx 10\%.$$

Similarly, suppose that the dividend news is bad this year and thus the dividend drops to \$0.95. Again, the Gordon formula implies that the stock price drops to $\frac{(1+1\%)\$0.95}{4\%-1\%} = \$31.98$, and therefore the stock return is

$$r_d = \frac{\$31.98 + \$0.95}{\$33.67} - 1 \approx -2\%.$$

As a result, the return volatility is $0.5 * (r_u - r_d) = 6\%$, which is the same as the dividend volatility.

What will happen if the risk attitude (and hence the equilibrium discount rate) becomes time varying? Let us suppose that with good news, the marginal investor becomes less risk averse so that the discount rate drops to 3.5% while with bad news, the investor becomes more risk averse and the discount rate rises to 4.5%. If this year the firm is doing well and issues a dividend of \$1.07, then the stock price jumps to $\frac{(1+1\%)\$1.07}{3.5\%-1\%} = \$43.23$, where the cash flow is discounted at the lower rate of 3.5%. So, the stock return with good news is

$$r_u = \frac{\$43.23 + \$1.07}{\$33.67} - 1 \approx 32\%.$$

Similarly, suppose that the dividend news is bad this year and the dividend drops to \$0.95. By the higher discount rate of 4.5%, the stock price then drops to $\frac{(1+1\%)\$0.95}{4.5\%-1\%} = \$27.41$ and therefore the stock return is

$$r_d = \frac{\$27.41 + \$0.95}{\$33.67} - 1 \approx -16\%.$$

Now the return volatility becomes $0.5 * (r_u - r_d) = 24\%$, which is much higher than the dividend volatility 6%.

The literature has used loss aversion to induce a changing risk aversion in two ways. First, Barberis et al. (2001) directly borrow another reasonable psychological assumption—the "house money effect"—and assume that losses are less painful to people if they occur after prior gains and more painful if they follow prior losses. Second, Li & Yang (2013a) show that this changing risk aversion can emerge endogenously because of a reversed disposition effect, which is ultimately driven by a time-varying, risk-aversion mechanism, as explained in Section 3.2. Underlying Li and Yang's theory is the existence of a discrepancy between trading frequency and evaluation frequency: although investors can trade over the course of a year, they evaluate investment performance only at the end of each year; when they retrade over the year, previous return realizations can change their location relative to the kink, which in turn shifts their risk attitude.

McQueen & Vorkink (2004) show that dynamic loss aversion also helps to explain volatility clustering found in low-frequency stock returns. They add a new feature to the framework developed by Barberis et al. (2001), namely, that after a large shock or several shocks of the same sign, investors not only become more or less risk averse, but they also become perturbed and thus more attentive to subsequent future news. This implies that any prominent news today will cause stock prices to react more to news tomorrow and hence future return volatility becomes higher, leading to volatility clustering in stock returns.

4.3. Return Predictability

Stock returns are forecastable. First, future stock returns can be predicted by past returns. On a six-month-to-one-year horizon, returns exhibit "momentum": stocks that have done well (poorly) over the past six months tend to keep doing well (poorly) over the next six months. In contrast, on a longer horizon, returns exhibit "reversal"; for instance, stocks that have done well (poorly) over the past three years tend to do poorly (well) in the future. Second, stock returns can be predicted by many other economic variables, such as price-dividend ratios, book-to-market ratios, and dividend volatility. For instance, the price-dividend ratio is able to explain more than 20% of the variation of NYSE stock returns; the "value premium" says that stocks with higher book-to-market ratios (i.e., value stocks) have higher average returns. Loss aversion is useful for understanding both types of return predictability through two approaches.

The first approach works through the time-varying, risk-aversion feature implied by dynamic loss aversion as we discussed earlier in Barberis et al. (2001) or in Li & Yang (2013a). For instance, in Barberis et al. (2001), investors become more risk averse after bad news, thus stock returns exhibit reversal—bad news pushes down prevailing prices and raises investors' risk aversion, leading to higher expected future returns as compensation for this raised risk aversion. The same mechanism also generates a value premium in the cross section: a stock with a low market-to-book ratio has often done badly in the past, accumulating prior losses for the investor, who then views it as riskier and requires a higher average return.

The second approach relies on an exogenous time-varying fundamental process to generate return predictability. For instance, Li & Yang (2013b) first document that dividends feature volatility clustering at both the aggregate level and the individual firm level, and then build this fact into a pricing model with loss-averse investors to show that dividend volatility is a robust predictor of future returns. Specifically, higher dividend volatility corresponds to a more volatile investment environment and hence more fluctuation in the investment values, which implies that potential losses faced by investors are now larger. This makes risky assets less desirable. As a result, investors may require more compensation when facing more volatile dividend processes, which, in turn, results in lower prices and higher equity premia.

4.4. What If Not All Investors are Loss Averse?

In previous applications of loss aversion on asset prices, researchers have typically assumed that all investors in the financial markets are loss averse. What happens if some investors are not? Answering this question is important because it can help us to qualify and quantify the pricing implications of loss aversion. Easley & Yang (2015) and Guo & He (2017) take on this task and examine the interactions between loss-averse investors and non-loss-averse investors. The key result of these studies is that the kink feature induced by loss aversion discourages loss-averse investors from holding risky assets, which significantly limits the pricing implications of loss aversion. Take the equity premium puzzle as an example. If all investors are loss averse, then the annual equity premium can be as high as 7%, as discussed above. However, if only half of investors exhibit loss aversion while the other half do not, then the equity premium may be only as low as 1%. In addition, even if loss-averse investors initially hold the majority of wealth so that the equity premium is initially very high, after a few years of trading, the equity premium can drop significantly because loss-averse investors are unwilling to participate in the equity market. These observations suggest that when we draw implications of loss aversion, we need to be careful about the heterogeneity of the investor base. For instance, retail investors are believed to be more loss averse than institutional investors thus the pricing implications discussed in the previous section may be more pronounced among stocks with lower institutional ownership.

5. CONCLUDING REMARKS

Over the past few decades, a sizable literature has emerged to analyze the implications of loss aversion for individual decisions and aggregate market outcomes. The literature is both theoretical and empirical and in this article, I have sought to synthesize some of the main themes and insights. I hope that this discussion will stimulate future research.

Loss aversion means that people perceive more disutility from losses than utility from equal-sized gains. The fundamental difficulty of applying loss aversion in finance is that defining gains and losses is very elusive and often context-specific. Still, researchers have made significant progress in drawing implications from loss aversion and found that it is useful in understanding investors' trading behavior, firm managers' real investment behavior, and the properties of stock returns. Generally speaking, loss aversion affects financial markets through changing market participants' risk attitudes: it makes people both globally and locally risk averse and induces risk attitudes to vary with prior gains and losses.

I make three main points in concluding this article. First, although loss aversion proves very useful in understanding financial markets, many ancillary assumptions are needed in order to build appropriate models. Such assumptions are crucial in deriving meaningful results. For example, it is the combination of narrow framing and loss aversion, not just loss aversion alone, that explains the nonparticipation puzzle and the equity premium puzzle. Thus, it remains important to build theories that shed light on the choice of these ancillary assumptions, in turn, presenting another great opportunity for research on loss aversion.

Second, most of the applications for market outcomes concern asset prices in a symmetric-information setting. It would be exciting to see whether loss aversion also delivers important implications for other aggregate outcomes, such as volumes or information efficiency in settings with asymmetric information. Li & Yang (2013a) have made an attempt to link loss aversion to price-volume dynamics. Pasquariello (2014) finds that loss aversion induces speculators to trade less or not at all with private information which may mitigate speculators' perceived risk of a trading loss. This result may have important implications for market efficiency and serve as a good starting point for future research on loss aversion in asymmetric-information settings.

Finally, some normative issues regarding loss aversion remain to be resolved and resolving these issues has important implications for market design. There are two dramatically different views that deliver contrasting policy implications. On the one hand, one can argue that loss aversion has some bias and can reduce people's welfare and thus with education, one can correct this bias and achieve a better welfare result. On the other hand, loss aversion may be innate, and how people behave reveals what they feel (prefer). Some economists even show that monkeys also exhibit loss aversion, suggesting that loss aversion is indeed a biological reaction, which may be a result of evolution. Further research is needed to advance our understanding of this dimension.

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